QUALIFYING EXAM COVER SHEET

January 2019 Qualifying Exams

Instructions: These exams will be “blind-graded” so under the student ID number please use your PUID

ID #: _________________________
(10 digit PUID)

EXAM (circle one)  514  519  523  530  544  553  554  562  571

For grader use:

Points _________ / Max Possible_________  Grade __________
MATH 544 QUALIFYING EXAMINATION
January 2019

Student Identifier: ____________________________________________

(PLEASE PRINT CLEARLY)

Instructions: There are a total of 6 problems in this exam. A problem appears on each of the following pages. Problems are worth 20 points each. Use the space provided for the solutions, using back pages as needed.
Problem 1 (20–pts) Let \((X, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space. Let \(f\) and \(g\) be nonnegative measurable functions with the property that

\[
\mu\{x \in X : g(x) > \lambda\} \leq \int_{\{x \in X : f(x) > \lambda\}} f(x) d\mu,
\]

for all \(\lambda > 0\). Prove that \(\int_X g^p d\mu \leq \int_X f^{p+1} d\mu\), for every \(0 < p < \infty\).
Problem 2 (20–pts) Consider $[0, 1]$ with its Lebesgue measure $m$. Suppose $\{f_k\}$ is a sequence of continuous functions on $[0, 1]$ such that $f_k \to f$ uniformly on $[0, 1]$ and $m\{x : f_k(x) < 0\} \to 0$, as $k \to \infty$. Prove that $f \geq 0$
Problem 3 (20–pts). Let \((X, \mathcal{F}, \mu)\) be a measure space and let \(\{f_n\}\) be a Cauchy sequence in \(L^1(\mu)\). Prove that for all \(\varepsilon > 0\) there is a \(\delta > 0\) such that for all \(n\),

\[
\int_E |f_n| \, d\mu < \varepsilon,
\]

whenever \(\mu(E) < \delta\).
Problem 4 (20–pts). Suppose \( \{f_n\} \) is a sequence of Lebesgue measurable functions on \([0, 1]\) with the property that every subsequence \( \{f_{n_k}\} \) has a further subsequence \( \{f_{n_{k,j}}\} \) such that \( f_{n_{k,j}}(x) \to 1 \), as \( j \to \infty \), for each \( x \in [0, 1] \). Prove that if \( |f_n(x)| \leq g(x) \), where \( g \in L^1[0, 1] \), then \( f_n \to 1 \) in \( L^1[0, 1] \), as \( n \to \infty \).
**Problem 5 (20–pts).** Let $m$ denote the Lebesgue measure. Let $\alpha > 1$. Prove that there exist $I_k \subset [0,1]$ such that if $\Omega = \bigcup_{k=1}^{\infty} I_k$, then

$$u(t) = \int_0^t \chi_\Omega dm > 0$$

for all $t > 0$ and

$$\frac{u(t)}{t^\alpha} \to 0$$

as $t \to 0^+$. Show that $u' = \chi_{\{u' \neq 0\}}$. 
Problem 6 (20–pts) Let $\beta > 1$. Prove that the limit

$$\lim_{k \to \infty} \int_0^k \left(1 + \frac{x}{k}\right)^k e^{-\beta x} \, dx$$

exists and find it. Justify all your steps.