QUALIFYING EXAM COVER SHEET

January 2019 Qualifying Exams

Instructions: These exams will be "blind-graded" so under the student ID number please use your PUID

ID #: ______________________________
(10 digit PUID)

EXAM (circle one) 514 519 523 530 **544** 553 554 562 571

For grader use:

Points __________ **/ Max Possible**__________ **Grade** __________

MATH 544 QUALIFYING EXAMINATION
January 2019

Student Identifier: ______________________________

(PLEASE PRINT CLEARLY)

Instructions: There are a total of 6 problems in this exam. A problem appears on each of the following pages. Problems are worth **20 points** each. Use the space provided for the solutions, using back pages as needed.

Problem 1 (20–pts) Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let f and g be nonnegative measurable functions with the property that

$$\mu\{x \in X : g(x) > \lambda\} \leq \int_{\{x \in X : f(x) > \lambda\}} f(x) d\mu,$$

for all $\lambda > 0$. Prove that $\int_X g^p d\mu \leq \int_X f^{p+1} d\mu$, for every $0 < p < \infty$.

Problem 2 (20–pts) Consider $[0, 1]$ with its Lebesgue measure m. Suppose $\{f_k\}$ is a sequence of continuous functions on $[0, 1]$ such that $f_k \to f$ uniformly on $[0, 1]$ and $m\{x : f_k(x) < 0\} \to 0$, as $k \to \infty$. Prove that $f \geq 0$

Problem 3 (20–pts). Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a Cauchy sequence in $L^1(\mu)$. Prove that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all n,

$$\int_E |f_n| d\mu < \varepsilon,$$

whenever $\mu(E) < \delta$.

Problem 4 (20–pts). Suppose $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0,1]$ with the property that every subsequence $\{f_{n_k}\}$ has a further subsequence $\{f_{n_{k_j}}\}$ such that $f_{n_{k_j}}(x) \to 1$, as $j \to \infty$, for each $x \in [0,1]$. Prove that if $|f_n(x)| \leq g(x)$, where $g \in L^1[0,1]$, then $f_n \to 1$ in $L^1[0,1]$, as $n \to \infty$.

Problem 5 (20–pts). Let m denote the Lebesgue measure. Let $\alpha > 1$. Prove that there exist $I_k \subset [0,1]$ such that if $\Omega = \cup_{k=1}^{\infty} I_k$, then

$$u(t) = \int_0^t \chi_\Omega dm > 0$$

for all $t > 0$ and

$$\frac{u(t)}{t^\alpha} \to 0$$

as $t \to 0^+$. Show that $u' = \chi_{\{u' \neq 0\}}$.

Problem 6 (**20–pts**) Let $\beta > 1$. Prove that the limit

$$\lim_{k\to\infty} \int_0^k (1+\frac{x}{k})^k e^{-\beta x} dx$$

exists and find it. Justify all your steps.