1. (18 points) Let X be a nonempty set, $\mathcal{A}$ a σ-algebra on X, and $\mu_n : \mathcal{A} \to [0, \infty]$, $n = 1, 2, \ldots$, a sequence of positive measures on $\mathcal{A}$ such that the limit $\mu(E) = \lim_{n\to\infty} \mu_n(E)$ exists for any $E \in \mathcal{A}$.

(a) Prove that if $\{\mu_n\}$ is increasing, i.e., $\mu_{n+1}(E) \geq \mu_n(E)$ for any $E \in \mathcal{A}$ and $n \in \mathbb{N}$, then μ is a measure on $\mathcal{A}$.

(b) Show by an example that if $\{\mu_n\}$ is decreasing, i.e., $\mu_{n+1}(E) \leq \mu_n(E)$ for any $E \in \mathcal{A}$ and $n \in \mathbb{N}$, then μ is not necessarily a measure on $\mathcal{A}$.

2. (18 points) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $A \in \mathcal{M}$ with $0 < \alpha \leq \mu(A) < \infty$, and $f : A \to \mathbb{R}$ a *strictly positive* function, integrable on A.

(a) Prove that

$$\inf_B \int_B f d\mu > 0,$$

where the infimum is taken over all measurable subsets $B \subset A$ with $\mu(B) \geq \alpha$.

(b) Show by an example that the statement of part (a) may fail if $\mu(A) = \infty$.

3. (18 points) Let $f \in L^1_{\text{loc}}(\mathbb{R})$. For a given number $\tau > 0$, we say that f is τ-periodic if

$$f(x+\tau) = f(x) \quad \text{for a.e. } x \in \mathbb{R}.$$

Prove that if there is a sequence of positive numbers $\{\tau_n\}$, such that f is τ_n-periodic for every $n = 1, 2, \ldots,$ and $\tau_n \to 0$ as $n \to \infty$, then there is constant $c \in \mathbb{R}$ such that $f(x) = c$ for a.e. $x \in \mathbb{R}$.

[*Note:* You may use without proof that the integrals of a τ-periodic function are constant over any interval of length τ.]

4. (18 points) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, $p > 1$, and $\{f_n\}_{n=1}^{\infty}$ a uniformly bounded sequence in $L^p(X, \mu)$, i.e., $\sup_n \|f_n\|_{L^p(X,\mu)} < \infty$. Prove that if f_n converges in measure to a function f on X, i.e.,

$$\lim_{n\to\infty} \mu\{|f_n - f| \geq \epsilon\} = 0, \quad \text{for any } \epsilon > 0,$$

then it converges to f in $L^r(X, \mu)$-norm for any $1 \leq r < p$, i.e.,

$$\lim_{n\to\infty} \|f_n - f\|_{L^r(X,\mu)} = 0.$$

5. (28 points) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, with m and M its minimal and maximal values on $[a, b]$, respectively. For $y \in [m, M]$, let $N(y)$ be the number of roots of the equation $f(x) = y$, $x \in [a, b]$, if that number is finite, and $N(y) = \infty$ if that number is infinite. (The function N is called the *Banach indicatrix* of f.)

Prove the following:

(a) The function N is Lebesgue measurable on $[m, M]$.

(b) The function N is Lebesgue integrable on $[m, M]$ if and only if f has a bounded variation on $[a, b]$. Moreover,

$$\int_m^M N(y)dy = T_f([a, b]).$$

(Here $T_f([a, b])$ denotes the total variation of the function f on $[a, b]$.)

[*Hint:* For $k \in \mathbb{N}$, let $I_1^{(k)} = [a, a + \frac{(b-a)}{2^k}]$, and $I_j^{(k)} = (a + (j-1)\frac{(b-a)}{2^k}, a + j\frac{(b-a)}{2^k}]$, $j = 2, 3, \ldots, 2^k$. For $y \in [m, M]$, define

$$L_j^{(k)}(y) = \chi_{f(I_j^{(k)})}(y) = \begin{cases} 1, & \text{if } y \in f(I_j^{(k)}), \\ 0, & \text{otherwise} \end{cases}$$

and

$$N^{(k)}(y) = L_1^{(k)}(y) + L_2^{(k)}(y) + \cdots + L_{2^k}^{(k)}(y).$$

Prove that $\{N^{(k)}(y)\}_{k=1}^{\infty}$ is a monotone sequence of functions, converging to $N(y)$ pointwise on $[m, M]$.]

[*Note:* You may use without proof that the image of an interval under a continuous function is an interval.]