QUALIFYING EXAM COVER SHEET

August 2022 Qualifying Exams

Instructions: These exams will be "blind-graded" so under the student ID number please use your PUID

ID #: ______________________________
(10 digit PUID)

EXAM (circle one) 530 **544** 553 554

For grader use:

Points __________ **/ Max Possible**__________ **Grade** __________

QUALIFYING EXAMINATION
August 2022
Math 544

Instructions: There are a total of 6 problems. A problem appears on each of the following pages. Problems are worth **20 points** each. Use the space provided for the solutions, using back pages as needed. Remember that for an interval $[a, b]$ with its Lebesgue measure m we simply write $\int_a^b f(x)dx$ for $\int_{[a,b]} f dm$, and similarly for open or half open intervals.

Write your solutions to each problem in clear, concise and correct English. Solutions must contain full details and should be presented clearly so that the grader can follow your argument.

Problem 1

(i) **(5–pts)** Define, carefully, what it means for a function $f : [0,1] \to \mathbb{R}$ to be of bounded variation.

(ii) **(5–pts)** Define, carefully, what it means for a function $f : [0,1] \to \mathbb{R}$ to be absolutely continuous.

(iii) **(10–pts)** Suppose $f \in L^p[0,1]$ and $g \in L^q[0,1]$, where $1 \leq p \leq \infty$ and q is its conjugate exponent. That is, $\frac{1}{p} + \frac{1}{q} = 1$. Set

$$F(x) = \int_0^x f(t)g(t)dt.$$

Prove that F is absolutely continuous on $[0,1]$.

(Extra page for work as needed)

Problem 2

(i) **(10–pts)** For $x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, consider the sequence $f_n(x) = \sum_{k=0}^{n} \sin^k(x)$ and compute (fully justifying all your steps)

$$\lim_{n\to\infty} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f_n(x)dx$$

(ii) **(10–pts)** Let $Q = \{q_k\}_{k=1}^{\infty}$ be **any** countable subset of $[0,1]$. Define the function f on $[0,1]$ by

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} 2^{-k} \frac{1}{\sqrt{|x-q_k|}}, & x \notin Q \\ \infty, & x \in Q \end{cases}$$

Prove that $\int_0^1 f(x)dx < \infty$.

(Extra page for work as needed)

Problem 3

(20–pts) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Suppose $\{f_n\}$ is a sequence of functions with $\int_X |f_n| d\mu = 5$ for all n. Suppose further that there exist measurable sets $E_1 \subset E_2 \ldots$ increasing to X (i.e., $\cup E_n = X$) such that $\int_{E_n} |f_n| d\mu \to 0$, as $n \to \infty$. Prove that the function $g(x) = \sup_n |f_n(x)| \notin L^1(\mu)$.

(Extra page for work as needed)

Problem 4

(20–pts) Suppose $(X, \mathcal{F}, \mu)$ is a measure space with $\mu(X) < \infty$. Let $\{c_k\}_{k=1}^{\infty}$ be (strictly) increasing sequence of positive numbers converging to infinity with the property that $\lambda c_{k+1} \leq c_k$ for all k for some $0 < \lambda < 1$. Prove that that a non-negative measurable function f belongs to $L^1(\mu)$ if and only if

$$\sum_{k=1}^{\infty} c_k \mu\{x \in X : c_k < f(x) \leq c_{k+1}\} < \infty.$$

(Extra page for work as needed)

Problem 5

(20–pts) Suppose f is measurable on a the σ-finite space $(X, \mathcal{F}, \mu)$ with the property that for all $r > 0$,

$$\mu\{x : |f(x)| > r\} \leq \frac{1}{r^2}.$$

Prove that

$$\int_E |f| d\mu \leq 2\sqrt{\mu(E)},$$

for all $E \in \mathcal{F}$.

(Extra page for work as needed)

Problem 6

(i) **(5–pts)** Let $(X, \mathcal{F}, \mu)$ be a measure space. Define convergence in measure.

(ii) **(15–pts)** For the two measurable functions f and g, define

$$\rho(f, g) = \int_X \frac{|f-g|^2}{1+|f-g|^2} d\mu.$$

Suppose $\mu(X) < \infty$. Prove that $f_n \to f$ in measure if and only if $\rho(f_n, f) \to 0$.

(Extra page for work as needed)