MA 544 Qualifying Exam, August 9, 2023

Each of the seven problems below is worth 5 points.

In your solutions make sure you justify your claims. Efforts to write neatly will be appreciated. The order of the problems is alphabetical, and is not intended to indicate their levels of difficulty.

No notes, books, cribsheets, or electronic devices, please.

1. Endow a set Ω with the counting measure ν: any $S \subset \Omega$ is measurable, and its measure is the number of elements it has, $0 \leq \nu(S) \leq \infty$. If $F : \Omega \to [-\infty, \infty]$ is integrable, prove that the set $\{x \in \Omega : F(x) \neq 0\}$ is at most countable.

2. For a map f between two metric spaces (M, ρ) and (P, σ) consider the following properties:

(1) f maps Cauchy sequences in M to Cauchy sequences in P.
(2) f is uniformly continuous.

Does (1) imply (2)? Does (2) imply (1)?

3. In a measure space $(\Omega, \mathcal{A}, \mu)$ we are given subsets $X_r \in \mathcal{A}$ for each rational r in such a way that $\mu(X_r \setminus X_s) = 0$ when $r \leq s$. Prove that there are $Y_a \in \mathcal{A}$ for all $a \in \mathbb{R}$ such that $Y_a \subset Y_b$ if $a \leq b$, and when r is rational, $X_r \subset Y_r$, $\mu(Y_r \setminus X_r) = 0$.

4. In this problem L^p refers to the L^p space of $[0, 1]$, endowed with Lebesgue measure. Suppose $1 \leq p < \infty$ and $g \in L^p$ has norm $||g||_p = 1$. Prove that there is a continuous linear $\Lambda : L^p \to \mathbb{R}$ of norm 1 such that $\Lambda(g) = 1$.

5. Suppose a compact metric space (M, ρ) has the property that for any pair $x, y \in M$ there is a $z \in M$ such that $\rho(x, z) = \rho(z, y) = \rho(x, y)/2$. Show that then there is also $u \in M$ such that $\rho(x, u) = \rho(x, y)/3$ and $\rho(u, y) = 2\rho(x, y)/3$.

6. Suppose $h : \mathbb{R}^2 \to \mathbb{R}$ and $H : \mathbb{R} \to [0, \infty)$ have the following properties. H is Lebesgue integrable; for any $x \in \mathbb{R}$ the function $h(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable; for any $t \in \mathbb{R}$ the function $h(\cdot, t) : \mathbb{R} \to \mathbb{R}$ is continuous; and $|h(x, t)| \leq H(t)$ for every $(x, t) \in \mathbb{R}^2$. Prove that the function $u : \mathbb{R} \to \mathbb{R}$,

$$u(x) = \int_{\mathbb{R}} h(x, t)\, dt \qquad \text{(Lebesgue integral)},$$

is continuous.

7. We are given a measure space $(\Omega, \mathcal{A}, \mu)$ and measurable functions $\phi_k : \Omega \to [-\infty, \infty]$, $k \in \mathbb{N}$, such that for every $\varepsilon > 0$ the series $\sum_k \mu\{x \in \Omega : |\phi_k(x)| > \varepsilon\}$ is convergent. Prove that for any $\delta > 0$ there is a set $E \subset \Omega$ of measure $< \delta$ such that ϕ_k converges uniformly on $\Omega \setminus E$.