MA 544 – Qualifying Exam— August 13, 2025

- **Do not open this booklet until you are told to do so.**
- **Put all phones and electronic devices (including earphones) away and do not handle them during the exam.**
- **The proctor will NOT answer math questions regarding the exam.**
- **This exam is about proofs, so you need to carefully justify your answers. If you quote a theorem, justify that its hypotheses are satisfied.**
- **No partial credit will be given for flawed arguments that achieve an apparent solution of a problem.**

- **Instructions:**

  1. The booklet has 8 pages, including this one.

  2. The number of points for each problem is written before its statement.

  3. Be organized and clearly show your work.

  4. Solve each problem on the corresponding space provided. You may use the back of the pages.

  5. Do not solve one problem on the space of another. If you need more paper, please ask. Do not use your own paper.

  6. If you do use extra paper and want me to grade what is written on it, turn it in and make sure you indicate which problem you are solving on each page. You should remove the staple and put pages in order. I have a stapler.

- **Academic Honesty**

  1. Do not seek or obtain any assistance from anyone to answer questions on this exam. Do not consult notes, books, etc. Do not handle electronic devices.

  2. A student who violates these rules will be given an F on the exam and the case will be referred to the Office of the Dean of Students.

1) (10 points) Use Hölder's inequality with suitable exponents to show that

$$\int_0^{\frac{\pi}{2}} x^{-\frac{1}{2}} \cos x \; dx \leq \left(\frac{16\sqrt{2\pi}}{3}\right)^{\frac{1}{3}}.$$

You may use that $\cos^3(x) = \frac{1}{4}(\cos(3x) + 3\cos x)$.

2) (10 points) Show that the following limit exists and compute its value

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} n^{\frac{3}{4}} \sinh\left(\frac{1}{n^{\frac{2}{3}}} \cos x\right)\, dx.$$

Recall that $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.

3) (10 points) Let $f, g \in L^1(\mathbb{R}^n)$. Let

$$F(x) = \int_{\mathbb{R}^n} |f(x-y)||g(y)| \; dy.$$

Show that $F(x) < \infty$ a.e.

4) (10 points) Let $g \in L^1(\mathbb{R})$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that $f, \frac{df}{dx} \in L^\infty(\mathbb{R})$. Define

$$H(x) = \int_{\mathbb{R}} f(x-y)g(y)\ dy \text{ (this is of course the Lebesgue integral).}$$

Prove that $H(x)$ is a well defined differentiable function on $\mathbb{R}$ and that

$$\frac{dH}{dx}(x) = \int_{\mathbb{R}} \frac{df}{dx}(x-y)g(y)\ dy.$$

5) Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a sequence $\{f_n : X \longrightarrow \mathbb{R}\}$ converges to f in measure if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mu\big(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}\big) = 0.$$

a)(5 points) Show that if $f_n \to f$ in $L^1(X)$, then $f_n \to f$ in measure.

b)(5 points) Is it always true that if $f_n \to f$ a.e, then $f_n \to f$ in measure?

c)(5 points) Is it true that a sequence f_n which converges in measure to a function f, also converges almost everywhere to f? Does it matter if $\mu(X) < \infty$ or not?

6) Let $(X, \mathcal{M}, \mu)$ be a measure space (μ is a positive measure)

a) (5 points) Show that if $\mu(X) < \infty$ and $1 < p < q \leq \infty$, then $L^q(X) \subset L^p(X)$.

b)(5 points) Suppose there exists a sequence of sets $\mathcal{O}_N \in \mathcal{M}$, $N \in \mathbb{N}$, such that

$$\lim_{N \to \infty} \mu(\mathcal{O}_N) = \infty.$$

Show that there exists a subsequence of sets $\mathcal{O}_{N_j} \in \mathcal{M}$, $\{\mathcal{O}_{N_j}\} \subset \{\mathcal{O}_N\}$, such that $\mu(\mathcal{O}_{N_j}) > 2^j$ and $\mu(\mathcal{O}_{N_{j+1}}) \geq 4\mu(\mathcal{O}_{N_j})$. If $E_{N_j} = \mathcal{O}_{N_j} \setminus \bigcup_{k=1}^{j-1} \mathcal{O}_{N_k}$ show that $E_{N_j} \cap E_{N_k} = \emptyset$ if $j \neq k$.

c) (10 points) Suppose $(X, \mathcal{M}, \mu)$ satisfies the assumptions of item b and let E_{N_j} be the family of sets defined there. For $p \in (1, \infty)$, let

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{\mu(E_{N_j})^{\frac{1}{p}}} \ \chi_{E_{N_j}}(x).$$

Show that $f \notin L^p(X)$ but $f \in L^q(X)$ for all $q \in (p, \infty)$.

7) Let $f, g : [0,1] \longrightarrow [0,1]$ be strictly decreasing continuous functions such that $f(0) = g(0) = 1$ and $f(1) = g(1) = 0$

a)(5 points) Show that f and g are invertible and their inverses f^{-1} and g^{-1} are strictly decreasing.

b)(20 points) Show that if the difference of the inverse functions $f^{-1} - g^{-1}$ is absolutely continuous and that for any interval $(a,b) \in (0,1)$,

$$m\big(\{x \in (0,\infty) : f(x) \in (a,b)\}\big) = m\big(\{x \in (0,\infty) : g(x) \in (a,b)\}\big),$$

where m denotes the Lebesgue measure, then $f(x) = g(x)$.