Purdue University

January 2025 Math 544 Qualifying Exam

Instructions

(i) Since the exams will be graded blindly, **please write your 10 digit PUID number:** ______________

(ii) There are two blank pages for each problem and additional scratch paper if you need it.

(iii) You have two hours to complete the exam. **Good luck!**

1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose $f : X \to \mathbb{R}$ satisfies

$$f^{-1}((r, \infty)) \in \mathcal{M}$$

for all $r \in \mathbb{Q}$. Prove that f is measurable.

2. Let $n \in \mathbb{N}$ be given, and suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz; that is, there exists C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in \mathbb{R}^n$ (with $|\cdot|$ denoting the usual Euclidean norm). Prove that if $E \subset \mathbb{R}^n$ has measure zero, then $f(E) \subset \mathbb{R}^n$ has measure zero.

3. Consider the Banach space $L^2([0,1])$ with the usual Lebesgue σ-algebra and measure. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be uniformly continuous. For $f \in L^2([0,1])$, define the function $Tf : [0,1] \to \mathbb{R}$ by

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy\,.$$

Suppose that $\{f_n\}_{n\in\mathbb{N}} \subset L^2([0,1])$ is a family of functions such that $\sup_{n\in\mathbb{N}} \|f_n\|_{L^2([0,1])} < \infty$. Prove that there is a subsequence of the family $\{Tf_n\}_{n\in\mathbb{N}}$ which converges uniformly on $[0,1]$.

4. Assume that $A \subset [0,1]$ is a Lebesgue-measurable set such that

$$0 < \mathrm{m}(A \cap I) < \mathrm{m}(I)$$

for every interval $I \subset [0,1]$. (You do not have to construct such a set). Let $F(x) := \mathrm{m}(A \cap [0,x])$. Prove that $F(x)$ is strictly increasing, absolutely continuous, but that $F'(x) = 0$ on a set of positive measure. *(Hint: Use the Lebesgue differentiation theorem.)*

5. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and f, $\{f_n\}_{n\in\mathbb{N}}$ be measurable complex-valued functions. Prove that the following are equivalent.

   (i) f_n converges to f in measure.

   (ii) For all subsequences $\{f_{n_k}\}_{k\in\mathbb{N}}$, there exists a further subsequence $\{f_{n_{k_j}}\}_{j\in\mathbb{N}}$ which converges almost uniformly.

6. Let $(X, \mathcal{M}, \mu)$ be a given measure space. Assume $\{f_k\}_{k\in\mathbb{N}}$ is an increasing sequence of measurable functions, $g \in L^1(\mu)$, and $f_k \geq g$ μ almost everywhere for all $k \in \mathbb{N}$. Prove that

$$\int_X f_k \,\mathrm{d}\mu := \int_{\{x\in X\,:\, f_k(x)>0\}} f_k \,\mathrm{d}\mu - \int_{\{x\in X\,:\, f_k(x)<0\}} -f_k \,\mathrm{d}\mu$$

belongs to $(-\infty, \infty]$ for each k and that

$$\lim_{k\to\infty} \int_X f_k \,\mathrm{d}\mu = \int_X \lim_{k\to\infty} f_k \,\mathrm{d}\mu\,.$$