In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide all details of your work.
(25 pts) 1). Show that the alternating subgroup $A_{11}$ of $S_{11}$ cannot have a subgroup of order $2,851,200 = 2^7 \cdot 3^4 \cdot 5^2 \cdot 11$.

(Hint: $A_{11}$ is simple. You need not prove this.)
2. Show that any group of order 294 is solvable.
3). (10 pts) a) Let $R$ be a non-zero commutative ring with 1. Show that if $I$ is an ideal of $R$ such that $1 + a$ is a unit in $R$ for all $a \in I$, then $I$ is contained in every maximal ideal of $R$.

(20 pts) b) Let $m \subset R$ be a unique maximal ideal. Then $a \in m$ if and only if $1 + ca$ is a unit.
4). Let $\phi : R \to S$ be a surjective homomorphism of commutative rings with $1 \neq 0$ and assume that $R$ contains a unique maximal ideal.

(10 pts) a) Show that $S$ contains a unique maximal ideal.

(10 pts) b) $\phi(1_R) = 1_S$.

(10 pts) c) Show that an element is a unit in $S$ if and only if it is the image of a certain unit in $R$.

(10 pts) d) Show that (b) is not true if $\phi : R \to S$ is not surjective.
5). Show that

(10 pts) a) \((x^2 + y^2)\) is irreducible in \(\mathbb{R}[x, y]\).

(10 pts) b) \(\mathbb{R}[x, y] / (x^2 + y^2)\) is an integral domain.
6). Determine the Galois groups of the following polynomials:

(10 pts) a) \( f(x) = x^3 - 2x + 4 \).

(10 pts) b) \( g(x) = x^3 - 3x + 1 \).

(10 pts) c) \( h(x) = f(x)g(x) \).
7). Let $\alpha = \sqrt[3]{1 + \sqrt{3}}$ and $\beta = \sqrt[3]{1 - \sqrt{3}}$

(5 pts) a) Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 6$

(10 pts) b) Prove that $K = \mathbb{Q}(\alpha, \beta, \sqrt{-1})$ is a normal closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$.

(10 pts) c) Show that $\sqrt[3]{2} \in K$.

(10 pts) d) Show that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{3}, \sqrt{-1})$, but $\omega \in \mathbb{Q}(\sqrt{3}, \sqrt{-1})$, where $\omega$ is a root of $\omega^2 + \omega + 1 = 0$. Conclude that $[L : \mathbb{Q}] = 12$, where $L = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}, \sqrt{-1})$. Thus $[K : \mathbb{Q}] = 12$ or 36.

(10 pts) e) Show that both $K/\mathbb{Q}$ and $L/\mathbb{Q}$ are extensions by radicals. Is $K/\mathbb{Q}$ solvable? Justify your answer.