QUALIFYING EXAM COVER SHEET

January 2020 Qualifying Exams

Instructions: These exams will be “blind-graded” so under the student ID number please use your PUID

ID #: __________________________
   (10 digit PUID)

EXAM (circle one)  514  519  523  530  544  553  554  562  571

For grader use:

Points ________ / Max Possible________ Grade ________
Instructions:

1. The point value of each exercise occurs adjacent to the problem.

2. No books or notes or calculators are allowed.

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1. (8 pts) Let \( g(x) \in F[x] \) be an irreducible monic polynomial of \( \deg n \geq 1 \), where \( F \) is a field of characteristic zero. Prove that \( g(x) \) does not have a multiple root.

2. (6 pts) Give an example of a field \( F \) having characteristic \( p > 0 \) and an irreducible monic polynomial \( g(x) \in F[x] \) that has a multiple root.

3. (6 pts) Give an example of a finite algebraic field extension \( L/F \) such that there exist infinitely many fields \( K \) with \( F \subseteq K \subseteq L \).
4. (5 pts) State Zorn’s Lemma.

5. (15 pts) Let $R$ be a commutative ring with $1 \neq 0$. Assume that $a \in R$ is such that $a^n \neq 0$ for each positive integer $n$, and let $S = \{a^n\}_{n \geq 0}$.

(i) Prove that there exists an ideal $I$ of $R$ such that $I$ is maximal among ideals of $R$ with $I \cap S = \emptyset$.

(ii) Prove that an ideal $I$ as in item (i) is a prime ideal.

(iii) Give an example of a ring $R$ having an element $a$ such that $a$ is a zero divisor and $a^n \neq 0$ for each positive integer $n$. 
6. (20 pts) Define what is meant by a composition series for a finite group $G$.

(a) State the Jordan-Hölder Theorem.

(b) Diagram the lattice of subgroups of the symmetric group $S_3$ and exhibit all the composition series for $S_3$. How many are there?

(c) Diagram the lattice of subgroups of the quaternion group $Q_8$ and exhibit all the composition series for $Q_8$. How many are there?

(d) How many composition series exist for the dihedral group $D_8$? Justify your answer.
7. (20) Let \( p \) be a prime number, and let \( \mathbb{F}_p \) denote the finite field with \( p \) elements.

(i) Prove that every finite algebraic extension field of \( \mathbb{F}_p \) is Galois.

(ii) Let \( K \) and \( L \) be finite algebraic field extensions of \( \mathbb{F}_p \).

(a) If \( [K : \mathbb{F}_p] = [L : \mathbb{F}_p] \), does it follow that \( K \) is isomorphic to \( L \)? Justify your answer.

(b) If \( [K : \mathbb{F}_p] \leq [L : \mathbb{F}_p] \), does it follow that \( K \) is isomorphic to a subfield of \( L \)? Justify your answer.

(iii) Let \( \overline{\mathbb{F}_p} \) denote the algebraic closure of \( \mathbb{F}_p \). If \( E \) is a subfield of \( \overline{\mathbb{F}_p} \) and \( [E : \mathbb{F}_p] = \infty \), does it follow that \( E \) is algebraically closed? Justify your answer.
8. Let $n$ and $p$ be positive integers with $p$ a prime integer. Let $Z = \langle x \rangle$ be a cyclic group of order $p^n - 1$.

(a) (7 pts) Describe the group $\text{Aut}(Z)$ of automorphism of $Z$. In particular, what is $|\text{Aut}(Z)|$?

(b) (7 pts) Let $\mathbb{F}_p$ be the field with $p$ elements and let $L/\mathbb{F}_p$ be a field extension of degree $n$. Let $G$ be the Galois group of $L/\mathbb{F}_p$. Describe the group $G$. In particular, what is $|G|$?

9. (6 pts) Let $G$ be a finite group and let $C$ be the center of $G$. If $G/C$ is abelian, does it follow that $C = G$? Justify your answer.
10. Let $L/F$ be a finite algebraic field extension.

(a) (10) If $L = F(\alpha)$ for some $\alpha \in L$, prove that there are only finitely many subfields $K$ of $L$ with $F \subseteq K$.

(b) (10) If there are only finitely many subfields $K$ of $L$ with $F \subseteq K$, prove that there exists an element $\alpha \in L$ such that $L = F(\alpha)$. 
11. (10 pts) Let $F$ be a field and let $F(x)$ denote the field of fractions of the polynomial ring $F[x]$. Let $\text{Aut } F(x)$ denote the group of automorphisms of the field $F(x)$, and let $\sigma \in \text{Aut } F(x)$ be such that $\sigma$ fixes $F$ and $\sigma x = x + 1$. Let $G = \langle \sigma \rangle$ be the cyclic subgroup of $\text{Aut } F(x)$ generated by $\sigma$.

(a) Depending on the characteristic of the field $F$, what is the order of the group $G$?

(b) Depending on the characteristic of the field $F$, give generators for the fixed field $F(x)^G$.

12. (10 pts) Let $p$ be a prime number and let $K/\mathbb{Q}$ be a splitting field of the polynomial $f(x) = x^p - 2 \in \mathbb{Q}[x]$.

(a) What is the degree of $K$ over $\mathbb{Q}$?

(b) Give generators for $K$ over $\mathbb{Q}$. 
13. (10 pts) Let $\alpha = \sqrt{2 + \sqrt{3}} \in \mathbb{R}$.

(a) What is the minimal polynomial for $\alpha$ over $\mathbb{Q}$?

(b) List the conjugates of $\alpha$ over $\mathbb{Q}$.

(c) List the conjugates of $\alpha$ over $\mathbb{Q}(\sqrt{2})$.

(d) Prove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois and describe the group $\text{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$.

14. (10 pts) Let $\beta = \sqrt{1 + \sqrt{3}} \in \mathbb{R}$.

(a) What is the minimal polynomial for $\beta$ over $\mathbb{Q}$?

(b) List the conjugates of $\beta$ over $\mathbb{Q}$.

(c) List the conjugates of $\beta$ over $\mathbb{Q}(\sqrt{3})$.

(d) Is $\mathbb{Q}(\beta)/\mathbb{Q}(\sqrt{3})$ Galois?

(e) Let $K$ be the splitting field of the minimal polynomial of $\beta$ over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$?
15. Let $K/\mathbb{Q}$ be the splitting field of the polynomial $x^4 + 1 \in \mathbb{Q}[x]$.

(a) (4 pts) What is the degree $[K : \mathbb{Q}]$?

(b) (8 pts) If $\alpha$ is one root of $x^4 + 1$, diagram the lattice of fields between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$, and give generators for each intermediate field.

16. (8 pts) True or false: If $f(x), g(x) \in \mathbb{Q}[x]$ are irreducible polynomials that have the same splitting field, then $\deg f = \deg g$. Justify your answer.
17. (20) Let $n > 1$ be a positive integer and let $p$ be a prime integer. Let $\varphi : \mathbb{Z}_{(pn)} \to \mathbb{Z}_{(n)}$ be the natural surjective ring homomorphism.

(a) If $p$ does not divide $n$ and $x$ is a unit in $\mathbb{Z}_{(n)}$, is every element in $\varphi^{-1}(x)$ a unit in $\mathbb{Z}_{(pn)}$? Justify your answer.

(b) If $p$ divides $n$ and $x$ is a unit in $\mathbb{Z}_{(n)}$, is every element in $\varphi^{-1}(x)$ a unit in $\mathbb{Z}_{(pn)}$? Justify your answer.

(c) Prove that $\varphi$ maps the units of $\mathbb{Z}_{(pn)}$ surjectively onto the units of $\mathbb{Z}_{(n)}$. 