Qualifying Exam for Math 553, 1994

1) (10 points) Prove or disprove: The multiplicative group of 2 x 2 matrices with integer coefficients and determinant 1 is generated by matrices of the following two forms,

$$1) \quad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

and

$$2) \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[Hint: Every square matrix can be diagonalized by elementary operations.]

2) (10 points) Let **G** be the symmetric group of a square (a dihedral group). Let v be a vertex, and **H**= Stab(v) (= the stablizer of v). Find orders o(**G**), o(**H**) and [**G**:**H**].

3) (10 points) Let p be a prime number. Prove or disprove that the center of a group of order p^3 can not be of order p^2.

4) (10 points) Prove or disprove that a group of order 77 must be cyclic.

5) (10 points) Find the prime factorization of $6 + 8i$ in the ring of Gaussian integers $\mathbf{Z}[i]$.

6) (10 points) (a) Prove that if $f(x) = x^p - x + a \in Z_p[x]$ where $Z_p = Z/pZ$ with p a prime number, then $f(x)$ is irreducible if and only if $a \neq 0$. [Hint: the above equation is invariant under the automorphisms x$\mapsto$ x+b for b=0, 1, 2, $\cdots$, $p-1$.]

(10 points) (b) Prove or disprove that $x^{101}+101x^{99}+202x^{30}-102x+25$ is irreducible in $\mathbf{Q}[x]$ where $\mathbf{Q}$ is the field of rational numbers.

7) (10 points) Let $\mathbf{R}$ be a finite field of characteristic p. Prove or disprove that every element in $\mathbf{R}$ has a p-th root.

8) (10 points) Let $\mathbf{Q}$ be the field of all rationals. Let $\mathbf{L}$ be a splitting field of the polynomial $x^9 + x^6 + x^3 + 1$. Find the Galois group of $\mathbf{L}$ over $\mathbf{Q}$.

9) (10 points) Let α be a root of the following equation over the field, $\mathbf{Q}$, of all rationals,

$$x^3 + x^2 - 2x - 1 = 0$$

Prove that $\mathbf{Q}[\alpha]$ is a Galois extension of $\mathbf{Q}$ and find the Galois group $\mathrm{Gal}(\mathbf{Q}[\alpha]/\mathbf{Q})$.
[Hint: $x^3+x^2-2x-1 = (x-\alpha)(x^2+(\alpha+1)x+1/\alpha) = (x-\alpha)(x^2+(\alpha+1)x+(\alpha^2+\alpha-2))$, and $-3\alpha^2 - 2\alpha + 9 = (2\alpha^2 + \alpha - 3)^2$.]