QUALIFYING EXAMINATION

AUGUST 1998

MATH 553 - PROFS. AVRAMOV/MOH

When answering any part of a problem you may assume the answers to the preceding parts.

The number of [points] carried by a correct answer is indicated after each question.

NOTATION: The letter p denotes a prime number.
The symbols $\mathbb{Z}$, $\mathbb{F}_q$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand for, respectively, the ring of integers, the field with q elements and those of rational, real, and complex numbers.

1. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial, such that p divides $f(n)$ for all $n \in \mathbb{Z}$. Prove that there exist polynomials $g(x), h(x) \in \mathbb{Z}[x]$ such that $f(x) = pg(x) + (x^p - x)h(x)$. [10]

2. Let $f(x) \in F[x]$ be an irreducible polynomial of degree $n \geq 2$ over a field F.
Let a and b be roots of $f(x)$ in some extension field K of F.

(1) Prove that a and b have the same order in the multiplicative group K^*. [5]
(2) Prove or disprove: This order is finite when $F = \mathbb{Q}$ and $[K : F] = 2$? [5]
(3) If the field F is finite, then show that this order is equal to the least integer $s \geq 0$ such that $f(x)$ divides the polynomial $x^s - 1$. [10]

3. Let $Z[i] = \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers.

(1) Write down a prime decomposition of $5 + 7i$ in $Z[i]$. [5]
(2) Prove that the factors in the chosen decomposition are prime elements. [10]

4. Let K be the splitting field of the polynomial $x^5 - 2$ over $F = \mathbb{Q}$.

(1) Find the degree $[K : F]$. [5]
(2) Describe the Galois group $G(K|F)$. [10]
(3) Find a primitive element for L over F. [5]

5. Prove that a group G of order 567 has a normal subgroup of order 27. [15]

6. Let G be a group of order $4n + 2$. Let G act on itself by left multiplication and let $\iota \colon G \to S_{4n+2}$ be the corresponding homomorphism to the symmetric group on $4n + 2$ elements.

(1) Prove that if $g \in G$ has order 2, then $\iota(g)$ is an odd permutation. [10]
(2) Prove that G contains a normal subgroup of order $2n + 1$. [10]