Math 553 **Qualifying Examination** **January, 1998**

Each problem 1–4 is worth 25 points. The points for individual parts are indicated by [–]. In working any part of a problem you may assume you have done the preceding parts.

1. [6] (a) In the group of symmetries of a regular hexagon (a group of order 12), how many elements are there of order 2? Of order 4? Of order 6?

[5] (b) Show that the multiplicative group generated by the complex matrices

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad (i^2 = -1,\ \omega^3 = 1,\ \omega \neq 1).$$

has order 12, and that it contains elements of orders 4 and 6.

[6] (c) (i) Prove that in the symmetric group S_4 the centralizer of a 3-cycle has order 3.

(ii) For a 3-cycle $c \in S_4$, how many distinct elements are there of the form aca^{-1} with a in the alternating group A_4. Why?

(iii) Prove that A_4 has no subgroup of order 6.

[8] (d) Altogether, how many different isomorphism classes are there of abelian groups of order 1200?

2. [6] (a) Show that a simple group which has a subgroup of index $n > 2$ is isomorphic to a subgroup of the alternating group A_n.

[4] (b) What is the smallest index $[A_n : G]$ occurring for a subgroup $G \subsetneqq A_n$? (Explain.)

[5] (c) Show that there is no simple group of order 112.

[5] (d) Show that there is no simple group of order 120.

<u>Hint</u>: Consider the normalizer of a Sylow 5-subgroup.

[5] (e) Is every group of order 120 solvable?

3. Let $\omega := (1 + \sqrt{-7})/2$.

[9] (a) Show that $\mathbb{Z}[\omega]$ is a euclidean domain.

[6] (b) Prove that 2 and 7 are not prime in $\mathbb{Z}[\omega]$.

[6] (c) Let $p \neq 7$ be an odd positive prime in $\mathbb{Z}$, let ζ be a primitive 7-th root of unity over the finite field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, and set $\xi := \zeta + \zeta^2 + \zeta^4$. Show that $(2\xi + 1)^2 = -7$, and that

$$-7 \text{ is a square in } \mathbb{F}_p \iff \xi^p = \xi \iff p \equiv 1,\ 2, \text{ or } 4 \pmod 7.$$

[4] (d) With p as in (c), show that p is prime in $\mathbb{Z}[\omega] \iff p \equiv 3,\ 5, \text{ or } 6 \pmod 7$.

4. For any integer n, set $c_n := 2\cos(2\pi n/7)$.

[7] (a) Find the minimal polynomial of c_1 over the rational field $\mathbb{Q}$, and determine its galois group. (Justify your answer.)

<u>Hint</u>: If $\zeta = e^{2\pi i/7}$ then $c_1 = \zeta + 1/\zeta$, and $\zeta^3 + \zeta^2 + \zeta + 1 + 1/\zeta + 1/\zeta^2 + 1/\zeta^3 = 0$.

[3] (b) Choose complex numbers s_1 and s_2 such that $s_i^2 = c_i$ $(i = 1, 2)$, and set $s_3 := 1/(s_1 s_2)$. Show that $s_3^2 = c_3$.

[5] (c) Show that every $\mathbb{Q}$-conjugate of $s := s_1 + s_2 + s_3$ is of the form $\pm s_1 \pm s_2 \pm s_3$, with either one or three + signs.

[10] (d) Find the minimal polynomial of s over $\mathbb{Q}$, and determine its galois group.