QUALIFYING EXAMINATION

JANUARY, 1999

MATH 553 - PROFS. AVRAMOV/LIPMAN

When answering any part of a problem you may assume the answers to the preceding parts.

The number of [points] carried by a correct answer is indicated after each question.

NOTATION: The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand for, respectively, the ring of integers and the fields of rational, real, and complex numbers.

1. Let G be a finite group, $p > 0$ a prime number, and H a normal p-subgroup of G. Prove the following assertions.

(1) H is contained in each Sylow p-subgroup of G. [5]

(2) If K is any normal p-subgroup of G, then HK is a normal p-subgroup of G. [5]

(3) The subgroup $O_p(G)$ generated by all normal p-subgroups of G is equal to the intersection of all the Sylow p-subgroups of G. [5]

(4) $O_p(G)$ is the unique largest normal p-subgroup of G. [5]

(5) $O_p(\overline{G}) = \{1\}$ where $\overline{G} = G/O_p(G)$. [5]

2. Let K be a field and let $p > 0$ be a prime number. Prove the following assertions.

(1) If $a, b \in K$ satisfy $a^n = b^p$ for some $0 < n < p$, then $a = c^p$ for some $c \in K$. [5]

(2) The polynomial $x^p - a$ is reducible in $K[x]$ if and only if $a = c^p$ for some $c \in K$. [5]

(3) If $K \subseteq \mathbb{R}$ and $\xi \in \mathbb{R}$ is such that $\xi^p \in K$, then an irreducible polynomial $f(x)$ of degree 3 in $K[x]$ is also irreducible in $K(\xi)[x]$. [10]

[HINT: After replacing K by $K(\sqrt{\Delta})$, where Δ is the discriminant of $f(x)$, one may assume that $\sqrt{\Delta} \in K$, and that the splitting field of K over F has degree 3.]

3. For a fixed negative integer $m \equiv 1 \mod (4)$ set $\mu = \dfrac{1+\sqrt{m}}{2}$ and $\overline{\mu} = \dfrac{1-\sqrt{m}}{2}$. Prove the following assertions.

(1) $\mathbb{Z}[\mu] = \{p + q\mu \in \mathbb{C} \mid p, q \in \mathbb{Z}\}$ is a ring and $\mathbb{Q}[\sqrt{m}] = \{s + t\sqrt{m} \in \mathbb{C} \mid s, t \in \mathbb{Q}\}$ is its field of fractions. [5]

(2) $\mathbb{Z}[\mu]$ is euclidean with respect to the norm

$$N(s + t\mu) = (s + t\mu)(s + t\overline{\mu}) = \left(s + \frac{t}{2}\right)^2 - m\left(\frac{t}{2}\right)^2$$

if and only if for all $s, t \in \mathbb{Q}$ there exist $p, q \in \mathbb{Z}$ with $N(s + t\mu - p - q\mu) < 1$. [10]

(3) $\mathbb{Z}[\mu]$ is euclidean for this norm if and only if $m = -3, -7, -11$. [5]

4. Let F be a finite field with q elements. Prove the following assertions.

(1) Polynomials $f(x), g(x) \in F[x]$ have the property that $\varphi(c) = g(c)$ for all $c \in F$ if and only if $g(x) \equiv f(x) \mod (x^q - x)$. [5]

(2) If $\varphi\colon F \to F$ is any map of sets, then there is a unique polynomial $f(x) \in F[x]$ of degree $\leq q-1$ such that $\varphi(c) = f(c)$ for all $c \in F$, namely

$$f(x) = \sum_{c \in F} \varphi(c)\big(1 - (x-c)^{q-1}\big)$$ [5]

(3) Fix $a \in F$ and make the convention that $0^0 = 1$. The polynomial of degree $\leq q-1$ corresponding to the map $\delta_a\colon F \to F$ given by $\delta_a(c) = \begin{cases} 1 & \text{if } c = a \\ 0 & \text{if } c \neq a \end{cases}$ is equal to

$$d_a(x) = 1 - \sum_{j=0}^{q-1} a^{q-1-j} x^j$$ [5]

(4) Elements $a_0, a_1, \ldots, a_{q-1}$ in F are pairwise distinct if and only if

$$\sum_{j=0}^{q-1} a_i^n = \begin{cases} 0 & \text{if } n = 0, 1, \ldots, q-2 \\ -1 & \text{if } n = q-1 \end{cases}$$ [5]

[HINT: Consider the map $\delta = \sum_{j=0}^{q-1} \delta_{a_i} : F \to F$.]