QUALIFYING EXAMINATION
January 2000
Math 553 – Prof. Goldberg

Instructions: Give a complete answer to each question. You may use any known result (be clear about what results you are using). When working part of a problem, you may assume the answer to the preceding parts.

1. (12 points) Find all groups of order $7 \cdot 11^3$ which have a cyclic subgroup of order 11^3.

2. Let R be a ring with identity 1 and consider the following two conditions:

(I) If $a, b \in R$ and $ab = 0$, then $ba = 0$;
(II) If $a, b \in R$ and $ab = 1$, then $ba = 1$;

(a) (10 points) Show that I implies II.

(b) (8 points) Show by example that II does not imply I.

3. Let F be a field. Suppose that E/F is a Galois extension, and that L/F is an algebraic extension with $L \cap E = F$. Let EL be the composite field, i.e., the subfield of an algebraic closure $\bar{F}$ of F generated by E and L.

(a) **(10 points)** Show EL/L is a Galois extension.

(b) **(8 points)** Show that there is an injective homomorphism

$$\varphi : \mathrm{Gal}(EL/L) \hookrightarrow \mathrm{Gal}(E/F).$$

Find the fixed field of the image of φ.

(c) **(6 points)** Show that $[EL : L] = [E : F]$.

(d) **(6 points)** Give an example to show that that the conclusion of (c) is false if we do not assume that E/F is Galois.

4. (12 points) Let G be a finite group. Let p be a prime and suppose that $|G| = p^k m$, with $k \geq 1$ and $p \not| m$. Let X be the collection of all <u>subsets</u> of G of order p^k. Then G acts on X by left multiplication, i.e., $g \cdot A = \{ga | a \in A\}$. For $A \in X$, denote by H_A the stabilizer in G of A. Show that $|H_A| \,\big|\, p^k$.

5. Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ be the ring consisting of polynomials with rational coefficients whose constant term is an integer.

(a) **(8 points)** Prove that R is an integral domain, with units ± 1.

(b) **(8 points)** Show that x is not an irreducible element of R.

(c) **(12 points)** Let $(x) = Rx$ be the ideal of R generated by x. Describe $R/(x)$ and show that $R/(x)$ is not an integral domain. What can you conclude about x?