Heinzer Math 553 Qualifying Exam 3 January 2001

Let $\mathbb{Z}$ denote the ring of integers and $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ the fields of rational, real and complex numbers, respectively.

(14) 1. Let k be a field and let $k(x)$ be the field of rational functions in x with coefficients from k. Let $t \in k(x)$ be the rational function $p(x)/q(x)$, where $p(x), q(x) \in k[x]$ are relatively prime polynomials and $q(x) \neq 0$. Assume that at least one of $p(x)$ and $q(x)$ is a nonconstant polynomial.

(i) Prove that $k(x)$ is algebraic over $k(t)$ and determine $[k(x) : k(t)]$.

(ii) Write down the minimal polynomial for x over the field $k(t)$.

(8) 2. Prove that an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ cannot have a multiple root.

(8) 3. Give an example of a field F having characteristic $p > 0$ and an irreducible polynomial $f(x) \in F[x]$ that has a multiple root.

(10) 4. Let K/F be an algebraic field extension. Suppose R is a subring of K such that $F \subseteq R$. Prove or disprove that R is a field.

(8) 5. Suppose $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$.

(i) Define "*α can be expressed by radicals* " or the equivalent phrase "*α can be solved for in terms of radicals.*"

(ii) For a polynomial $f(x) \in \mathbb{Q}[x]$, define "*$f(x)$ can be solved by radicals.*"

(14) 6. Let G be the Galois group of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$, where $\deg f = 5$.

(i) What integers are possible for the order of G? Explain your answer.

(ii) If G contains an element of order 3, what integers are possible for the order of G? Explain your answer.

(20) 7. Let $\omega \in \mathbb{C}$ be a primitive 12-th root of unity.

(i) What is $[\mathbb{Q}(\omega) : \mathbb{Q}]$?

(ii) List the distinct conjugates of $\omega + \omega^{-1}$ over $\mathbb{Q}$.

(iii) What is the group $\mathrm{Aut}(\mathbb{Q}(\omega + \omega^{-1})/\mathbb{Q})$? Is $\mathbb{Q}(\omega + \omega^{-1})$ Galois over $\mathbb{Q}$?

(iv) Diagram the lattice of subfields of $\mathbb{Q}(\omega)$ giving generators for each.

(14) 8. Diagram the lattice of ideals of the ring $\mathbb{Z}[x]/(15, x^3 + 1)$.

(8) 9. Let $f(x) \in \mathbb{Q}[x]$ be a monic polynomial of degree n. Define the *discriminant* of $f(x)$.

(18) 10. Let p be a prime. Recall that a field extension K/F is called a p-extension if K/F is Galois and $[K : F]$ is a power of p.

(i) Suppose K/F and L/K are p-extensions. Prove that the Galois closure of L/F is a p-extension.

(ii) Give an example where K/F and L/K are p-extensions, but L/F is not Galois.

(16) 11. Let K/F be a finite separable algebraic field extension and let $\alpha \in K$.

(i) Define the norm $N_{K/F}(\alpha)$ of α from K to F.

(ii) Prove that $N_{K/F}(\alpha) \in F$.

(iii) Define the trace $Tr_{K/F}(\alpha)$ of α from K to F.

(iv) Prove that $Tr_{K/F}(\alpha) \in F$.

(8) 12. Let $\mathbb{F}_5$ denote the field with 5 elements. What is the order of the group $\mathrm{SL}_2(\mathbb{F}_5)$ of 2×2 matrices with entries in $\mathbb{F}_5$ that have determinant 1?

(10) 13. Suppose R is an integral domain in which each prime ideal is a principal ideal. Prove or disprove that every ideal of R is principal.

(10) 14. Let R be a commutative ring with identity $1 \neq 0$ and let $f(x), g(x) \in R[x]$ be polynomials. Let $c(f), c(g)$ denote the ideals of R generated by the coefficients of $f(x), g(x)$, respectively. If $c(f) = c(g) = R$, prove that the ideal $c(fg)$ generated by the coefficients of the product $f(x)g(x)$ is also equal to R.

(14) 15. Suppose $L/\mathbb{Q}$ is a finite algebraic field extension for which there exists a chain

$$\mathbb{Q} = L_0 \subset L_1 \subset \cdots \subset L_{n-1} \subset L_n = L,$$

where $[L_{i+1} : L_i] = 2$, for each $i = 0, \ldots n-1$. If K is a subfield of L, prove or disprove that there exists for some integer $m \leq n$ a chain

$$\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_{m-1} \subset K_m = K.$$

where $[K_{i+1} : K_i] = 2$, for each $i = 0, \ldots, m-1$.

(12) 16. Let G be a finite group with $|G| = n$.

(i) Prove that G is isomorphic to a subgroup of the symmetric group S_n.

(ii) Is G isomorphic to a subgroup of the alternating group A_m for some positive integer m? Justify your answer.

(8) 17. Suppose H and K are normal subgroups of a group G and that $H \cap K = 1$. Prove or disprove that for each $x \in H$ and $y \in K$, we have $xy = yx$.