QUALIFYING EXAMINATION
JANUARY 2003
MATH 553 - Prof. Moh

1) (10 points) Let GL(2,$\mathbf{R}$) be the multiplicative group of all non-singular 2×2 matrices with real entries. Let the group GL(2,$\mathbf{R}$) acts from left on $\mathbf{R}\times$ $\mathbf{R}$. Find all orbits.

2) (10 points) (A) Find a commutative ring without any maximal ideal. (B) Prove or disprove: a group $\mathbf{G}$ must be commutative if there is a commutative normal subgroup $\mathbf{H}$ such that $\mathbf{G}/$ $\mathbf{H}$ is commutative.

3) (10 points) Let $\mathbf{R}$ be a commutative ring and $f(x) \in \mathbf{R}[\mathrm{x}]$ is nilpotent. Show that there is an element $0 \neq a \in \mathbf{R}$ such that $af(x) = 0$.

4) (10 points) Let p, q be positive prime numbers. Show that any group of order p^2q is not simple.

5) (10 points) Find a greatest common divisor of $7+i$ and $5+9i$ in the ring of Gaussian integers $\mathbf{Z}[i]$.

6) (10 points) In the group $\mathbf{GL(2,R)}$, let

$$g_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Find $o(g_1), o(g_2), o(g_1 g_2)$.

7) (10 points) Let $\mathbf{R}$ be a finite field, and n any positive integer. Show that there is an irreducible polynomial of degree n in $\mathbf{R}[\mathrm{x}]$.

8) (10 points) Let $\mathbf{R}$ be a non-commutative ring. Show that if $1 - ab$ is invertible for some elements $a, b \in \mathbf{R}$, then $1 - ba$ is invertible $\in \mathbf{R}$.

9) (10 points) Let **L** be the splitting field of the polynomial $x^4 - 2$ over **Q** . Find the Galois group of **L** over **Q**.

10) (10 points) Let **K** be a finite field of characteristic p. Show that $x^p - x - a$ for some element $a \in \mathbf{K}$ is either irreducible or can be factored into a product of linear polynomials.