QUALIFYING EXAMINATION
AUGUST 2004
MATH 553 - Prof. Heinzer

Let $\mathbb{Z}$ denote the ring of integers and $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ the fields of rational, real and complex numbers, respectively.

(15) 1. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

(i) Let $\mathrm{Aut}(\overline{\mathbb{Q}})$ denote the group of automorphisms of $\overline{\mathbb{Q}}$. Is the group $\mathrm{Aut}(\overline{\mathbb{Q}})$ finite or infinite? Justify your answer.

(ii) Is the group $\mathrm{Aut}(\overline{\mathbb{Q}})$ abelian? Justify your answer.

(iii) Is the group $\mathrm{Aut}(\mathbb{R})$ of automorphisms of $\mathbb{R}$ abelian? Justify your answer.

(18) 1. (continued)

(iv) Prove the existence of a subfield K of $\overline{\mathbb{Q}}$ that is maximal with respect to $\sqrt[3]{2} \notin K$.

(v) Prove the existence of subfields K_1 and K_2 of $\overline{\mathbb{Q}}$ each maximal with respect to not containing $\sqrt[3]{2}$ such that $[K_1(\sqrt[3]{2}) : K_1] = 3$ and $[K_2(\sqrt[3]{2}) : K_2] = 2$.

(vi) With K as in (iv), if L/K is a finite algebraic field extension, prove that L/K is cyclic and that either $[L : K] = 3^n$ or $[L : K] = 2^n$ for some $n \in \mathbb{N}$.

Recall that if R and S are rings, then $R \times S = \{(r, s) \,|\, r \in R, s \in S\}$ is a ring where addition and multiplication in $R \times S$ are defined componentwise.

(6) 2. Describe all the prime ideals of $\mathbb{Z} \times \mathbb{Z}$.

(12) 3. Consider the polynomial ring $\mathbb{Z}[x]$.

(i) Define $\phi_1 : \mathbb{Z}[x] \to \mathbb{Z}$, by $\phi_1(f(x)) = f(1)$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi_1$.

(ii) Define $\phi_2 : \mathbb{Z}[x] \to \mathbb{Z}$, by $\phi(f(x)) = f(2)$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi_2$.

(iii) Define $\phi : \mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z}$ by $\phi(f(x)) = (f(1), f(2))$ for every $f(x) \in \mathbb{Z}[x]$. Give generator(s) for the ideal $\ker \phi$.

(iv) Prove or disprove that ϕ is surjective.

(18) 4. Let K/F be a finite separable algebraic field extension and let $\alpha \in K$.

(i) Define the norm $N_{K/F}(\alpha)$ of α from K to F.

(ii) Prove that $N_{K/F}(\alpha) \in F$.

(iii) Define the trace $Tr_{K/F}(\alpha)$ of α from K to F.

(iv) Prove that $Tr_{K/F}(\alpha) \in F$.

(v) For $K = \mathbb{Q}(\sqrt[3]{2})$, compute $N_{K/\mathbb{Q}}(\sqrt[3]{2})$ and $Tr_{K/\mathbb{Q}}(\sqrt[3]{2})$.

(vi) For $K = \mathbb{Q}(\sqrt[3]{2})$, compute $N_{K/\mathbb{Q}}(3)$ and $Tr_{K/\mathbb{Q}}(3)$.

(7) 5. True or false: If $f(x), g(x) \in \mathbb{Q}[x]$ are irreducible polynomials that have the same splitting field, then $\deg f = \deg g$. Justify your answer.

(14) 6. (i) Give generators for each maximal ideal of the polynomial ring $\mathbb{Z}[x]$ that contains the ideal $(15, x^2 - 3)$.

(ii) Diagram the lattice of ideals of the polynomial ring $\mathbb{Z}[x]$ that contain the ideal $(15, x^2 - 3)$.

(6) 7. Suppose $L/\mathbb{Q}$ is a finite algebraic field extension with $[L : \mathbb{Q}] = 4$. Is it possible that there exist exactly two subfields K_1 and K_2 of L for which $[L : K_i] = 2$? Justify your answer.

(6) 8. Suppose $L/\mathbb{Q}$ is a finite algebraic field extension with $[L : \mathbb{Q}] = 4$. Is it possible that there does not exist a subfield K of L for which $[L : K] = 2$? Justify your answer.

(16) 9. Let $\omega \in \mathbb{C}$ be a primitive 12-th root of unity.

(i) What is $[\mathbb{Q}(\omega) : \mathbb{Q}]$?

(ii) List the distinct conjugates of $\omega + \omega^{-1}$ over $\mathbb{Q}$.

(iii) What is the group $\mathrm{Aut}(\mathbb{Q}(\omega + \omega^{-1})/\mathbb{Q})$? Is $\mathbb{Q}(\omega + \omega^{-1})$ Galois over $\mathbb{Q}$?

(iv) Diagram the lattice of subfields of $\mathbb{Q}(\omega)$ giving generators for each.

(12) 10. Let F be a field. For each nonconstant monic polynomial $f = f(x) \in F[x]$, let x_f be an indeterminate. Consider the polynomial ring $R = F[\{x_f\}]$, and let I be the ideal of R generated by the polynomials $f(x_f)$, where f varies over all the nonconstant monic polynomials in $F[x]$.

(i) Prove that $I \neq R$.

(ii) Prove that there exists an extension field K of F in which each nonconstant monic polynomial $f \in F[x]$ has a root.

(8) 11. Is $\mathbb{Q}(\sqrt{2})$ the fixed field of an automorphism of $\overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$? Justify your answer.

(16) 12. Let $K/\mathbb{Q}$ be the splitting field of the polynomial $x^9 - 1 \in \mathbb{Q}[x]$.

(i) Diagram the lattice of subfields of $K/\mathbb{Q}$. For each subfield, give generators and list its degree over $\mathbb{Q}$.

(ii) Let $\alpha \in \mathbb{R}$ be a root of the polynomial $x^9 - 2 \in \mathbb{Q}[x]$ and let L be the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. What is $[L : \mathbb{Q}(\alpha)]$?

(iii) Diagram the lattice of subfields F of L that contain $\mathbb{Q}(\alpha)$. For each such field F, list $[F : \mathbb{Q}(\alpha)]$ and give a generator for F over $\mathbb{Q}(\alpha)$.

(7) 13. Suppose $f(x) \in \mathbb{Q}[x]$ is a monic polynomial of degree n and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ are the roots of $f(x)$. Let G be the Galois group of $f(x)$ over $\mathbb{Q}$. Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$ if and only if the action of G on $\{\alpha_1, \dots, \alpha_n\}$ is transitive.

(7) 14. Let G act as a transitive permutation group on the finite set A with $|A| > 1$. Prove that there exists $g \in G$ such that $g(a) \neq a$ for all $a \in A$.

(16) 15. Let R be an integral domain.

(i) Define R is a *unique factorization domain* (UFD).

(ii) Give an example of an integral domain that is not a UFD.

(iii) Assume that R is a subring of a UFD S and that every unit of S is contained in R. Also assume for all a and b in S, if $ab \in R$, then a and b are in R. Prove that R is a UFD.

(iv) If the polynomial ring $R[x]$ is a UFD, prove that R is a UFD.

(8) 16. Let R be a commutative ring with identity $1 \neq 0$ and let $f(x)$ and $g(x)$ be polynomials in $R[x]$. Let $c(f), c(g)$ denote the ideals of R generated by the coefficients of $f(x), g(x)$, respectively. If $c(f) = c(g) = R$, prove that the ideal $c(fg)$ generated by the coefficients of the product $f(x)g(x)$ is also equal to R.

(8) 17. Assume that F is a field of characteristic zero and that K/F is an algebraic field extension. If each nonconstant polynomial in $F[x]$ has at least one root in K, prove that K is algebraically closed.