QUALIFYING EXAMINATION
JANUARY 2006
MATH 553 - Prof. Kim

You must justify all assertions for full credit. A good understanding of what level of justification is reasonable should be regarded as an integral part of the exam. So, while an elaborate proof will sometimes be appropriate, there will be other times when a quick statement is preferrable to a long sequence of formulas expounding on a simple point. The totality *of understanding expressed by the solution is what will be graded.*

1. (20 pts) Let $M_3(\mathbb{Z})$ be the ring of 3 by 3 matrices with integer entries. Let

$$h : \mathbb{Z}[x] \rightarrow M_3(\mathbb{Z})$$

be the ring homomorphism that sends x to the matrix

$$T = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Determine the kernel of h.

2. (10 pts) Let A be the abelian group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and let $G = Aut(A)$ be the group of automorphisms of A.
(a) Determine the order of G.
(b) Give a precise description of the center of the group G.
(c) Describe a 3-Sylow subgroup of G.
(d) For the subgroup in (c) describe its normalizer.
(e) Determine the number of 3-Sylow subgroups of G.

3. (10 pts) (a) Find all simple groups of order 101.
(b) Find all simple groups of order 102.
(c) Find all groups of order 175 (Note that here, we are not asking for the groups to be simple).

4. (20 pts) Let $R := \mathbb{Q}[t]/(t^2)$.
(a) Describe fully the group G of automorphisms of R when it is regarded only as an abelian group.
(b) Describe the group H of automorphisms of R compatible with its ring structure.
(c) Repeat part (a) for $S = \mathbb{Q}[t]/(t^2 - 3t + 2)$.
(d) Repeat part (b) for $S = \mathbb{Q}[t]/(t^2 - 3t + 2)$.

5. (10 pts)
(a) Is $\mathbb{C}[x, y]/(x^2 - y^3)$ an integral domain?
(b) Is $\mathbb{C}[x, y]/(x^2 - y^3)$ a unique factorization domain? (Hint: Use the subring R of $\mathbb{C}[t]$ generated by t^2 and t^3. In order to use this hint correctly, you need to be a bit careful.)
(c) Is $\mathbb{Z}[x, y]/(x^3 + yx^2 + xy^2 - 2xy - 3x^2 + y^2 - 3x - 9)$ an integral domain?

6. (10 pts) Let K be a field and let G be a finite group acting on K as field automorphisms. Denote by

$$F := \{x \in K | gx = x, \forall g \in G\}$$

the fixed field of G.
(a) Show that if an irreducible polynomial $f \in F[x]$ has a root in K, then it factors into linear terms in $K[x]$.
(b) Suppose now that K is a subfield of the algebraic numbers $\bar{\mathbb{Q}}$. Use part (a) to show that every automorphism in $Aut(\bar{\mathbb{Q}}/F)$ stabilizes K.

(c) Find a counterexample to (b) in the following sense: Find some tower of extensions

$$\begin{array}{c} L \\ | \\ K \\ | \\ F \end{array}$$

and an element of $Aut(L/F)$ that does *not* stablize K. (Note here that F should not be the fixed field of a finite group of automorphisms of K.)

7. (20 pts) Find the Galois group over $\mathbb{Q}$ of the polynomial $x^{10} - 4x^8 + 2x^6 - x^4 + 4x^2 - 2$.