MATH 553 QUALIFYING EXAMINATION
January 6, 2012

PLEASE BEGIN YOUR ANSWER TO EACH PROBLEM I—IV ON A NEW SHEET.

WHEN ANSWERING ANY PART OF A PROBLEM, YOU MAY USE PRECEDING PARTS, EVEN IF YOU HAVEN'T SOLVED THEM.

I. [15 points] Let G be a finite group, let p be a prime divisor of the order $|G|$, and let P be a Sylow p-subgroup of G. Let $N(P)$ be the normalizer of P, and $C(P) \subset N(P)$ the centralizer of P (consisting of those elements of G which commute with every element of P).

(a) Show that the index $[N(P) : C(P)]$ is the order of a subgroup of the automorphism group of P.

(b) Show that p divides $[N(P) : C(P)]$ if, and only if, P is nonabelian.

(c) Show that if P is cyclic and the gcd $\big(|G|, p-1\big) = 1$ then $C(P) = N(P)$.

II. [15 points] (a) Let R be an integral domain. Suppose there exists a function $\nu\colon \big(R\setminus\{0\}\big) \to \mathbb{N}$ such that ($*$): for all $b \neq 0$ in R and $a \notin bR$, there exist x and y in R such that $ax + by \neq 0$ and $\nu(ax + by) < \nu(b)$. Prove that R is a principal ideal domain.

(b). Let R be any principal ideal domain. Define $\nu\colon \big(R \setminus \{0\}\big) \to \mathbb{N}$ by $\nu(a) := 0$ if a is a unit, and otherwise $\nu(a) :=$ the total number of factors in any factorization of a into irreducible elements.

(i) Explain why this ν is well-defined.

(ii) Show that ν satisfies the condition ($*$) in (a).

III. [10 points] Let F be a finite field of odd cardinality q, and let $L \supset F$ be an extension such that $[L : F] = 2$.

(a) Show that the roots in L of $X^8 - 1$ form a cyclic group G of order 8.

Hint. Consider the order of the multiplicative group L^*.

(b) Let ζ be a generator of G. Show that $\big(\zeta + \zeta^{-1}\big)^2 = 2$.

(c) Show that $\zeta + \zeta^{-1} \in F \iff \big(\zeta + \zeta^{-1}\big)^q = \zeta + \zeta^{-1} \iff q \equiv \pm 1 \pmod 8$.

IV. [20 points] (a) Prove: the galois group $\mathcal{G}$ of $f(X) := (X^3 - 2)(X^3 - 3) \in \mathbb{Q}[X]$ is isomorphic to the semi-direct product $\mathbb{Z}_2 \ltimes_\theta (\mathbb{Z}_3 \times \mathbb{Z}_3)$, where θ takes the generator of $\mathbb{Z}_2$ to the automorphism $h \mapsto h^{-1}$ of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) Set $\rho := e^{2\pi i/3}$, so that $L := \mathbb{Q}[\rho, \sqrt[3]{2}, \sqrt[3]{3}\,]$ is a splitting field of $f(X)$. ($\sqrt[3]{\ }$ means *real* cube root.) Describe explicitly all the subfields of L that contain ρ. (One such field, for example, is $\mathbb{Q}[\rho, \sqrt[3]{12}\,]$.)

`Suggestions (fill in—and justify—the details!):`

For (a), let $E := \mathbb{Q}[\rho] \subset \mathbb{Q}[\rho, \sqrt[3]{2}\,] =: F$. Show that $X^3 - 2$ is irreducible in $E[X]$. If $X^3 - 3$ were reducible in $F[X]$ then for some $d \in F$, say $d = a + b\sqrt[3]{2} + c\sqrt[3]{4}$ $(a, b, c \in E)$, we'd have $d^3 = 3$, and so if ψ is the E-automorphism of F such that $\psi\sqrt[3]{2} = \rho\sqrt[3]{2}$ then $\psi(d) = \rho^n d$ for some $n = 1$ or 2; deduce from this that a and one of b, c vanish, then take the $(E/\mathbb{Q})$-norm of d^3 to get a contradiction. Conclude that $[L : E] = 9$, and determine the galois group H of L/E.

Note also that complex conjugation is an automorphism of L, generating an order 2 subgroup $G \subset \mathcal{G}$.

For (b), think about the subgroups of H.