FALL 2014

Qualifying Exam - MA 553

In answering any part of a question, you may assume the results in previous parts, even if you have not solved them. Be sure to provide all details of your work.

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1. Show that P is solvable if $\mathrm{Aut}(P)$ is abelian. (10 points)

   Is converse true? (10 points)

2. Show that any group of order 405 is solvable. (20 points)

3. Let $\phi : G \to H$ be a epimomorphism of groups.

   (a) Show that the image $\phi(P_G)$ of any Sylow p-subgroup P_G of G is a Sylow p-subgroup of H. (10 points)

   (b) Show that any Sylow p-subgroup P_H of H is the image $\phi(P_G)$ of a certain Sylow p-subgroup P_G of G. (10 points)

4. Find all (up to isomorphism) abelian groups of order 500. (10 points)
   Find the number of elements of order 2 in each of them. (10 points)

5. Let R be an integral domain and F its field of fractions. Let P be a prime ideal in R and $R_P = \{\frac{a}{b} \mid a, b \in R, b \notin P\} \subset F$.

   Show that R_P has a unique maximal ideal. (20 points)

6. Let $(R, +,)$ be a commutative ring with $1 \neq 0$. Show that the following conditions are equivalent:

   (a) R has a unique maximal ideal M

   (b) The set of non-units in R is an ideal. (20 pts)

7. Find a simpler description for each of the following rings.

   (a) (10 pts) $Q[x]/(x^3 + x)$.

   (b) (10 pts) $Z[x]/(x - 2, x^2 + 1)$.

8. Consider the polynomial $f(x) = x^4 + 2x^2 + 4$ over Q.

   (a) Express all roots in terms of radicals. (5 points)

   (b) Determine the Galois group of the splitting field $L = Q_f$ over Q. (15 pts)

9. Let K be the splitting field of $(x^3+x^2+1)(x^3+x+1)(x^2+x+1)$ over F_2. Find the Galois group $Gal(K/F_2)$ and its generator(s).(15 points) Find all intermediate fields $F_2 \subseteq F \subseteq K$. (5 points)

10. Let n be a positive integer and d a positive integer that divides n. Suppose $a \in R$ is a root of the polynomial $x^n - 7 \in Q[x]$. Prove that there is precisely one subfield F of $Q(a)$ with $[F : Q] = d$. (20 pts)