QUALIFYING EXAMINATION

August 2016

MA 553

1. (14 points) Determine all groups of order 255 up to isomorphism. (Hint: Show that the center is not trivial.)

2. (13 points) Let G be a group of order 375 that has a cyclic 5-Sylow subgroup. Show that G is cyclic.

3. (16 points) We write $\mathbb{Z}[i] \subset \mathbb{C}$ for the ring of Gaussian integers and $\mathbb{Z}[X]$ for the polynomial ring in one variable.

(a) Show that there is an isomorphism of rings $\mathbb{Z}[i] \cong \mathbb{Z}[X]/(X^2+1)$.

(b) Determine the number of distinct ideals of the ring $\mathbb{Z}[i]$ that contain $3+i$.

4. (14 points) Let k be a field, f an irreducible polynomial in $k[X]$, and $k \subset K$ a normal field extension. Consider a factorization $f = f_1 \cdots f_s$, where f_i are irreducible polynomials in $K[X]$. Show that all f_i have the same degree.

5. (17 points) Let K be a field of characteristic zero with algebraic closure $\overline{K}$, and let α be an element of $\overline{K}$. Let L be a maximal intermediate field of $K \subset \overline{K}$ not containing α, i.e., a subfield of $\overline{K}$ with $K \subset L$ and $\alpha \notin L$ so that $L = L'$ for every subfield L' of $\overline{K}$ with $L \subset L'$ and $\alpha \notin L'$. Show that the field extension $L \subset \overline{K}$ is Galois and Abelian. (Hint: First show that every finite Galois extension of L in $\overline{K}$ is cyclic.)

6. (14 points) Let $k \subset K$ be a finite Galois extension and L an intermediate field, $k \subset L \subset K$, so that $[K:k]$ does not divide $[L:k]\,!$. Show that there exists an intermediate field L', $L \subset L' \subset K$, so that $L' \neq K$ and $k \subset L'$ is Galois.

7. (12 points) Let p be a prime number and let $K \subset \mathbb{C}$ be the spitting field of $X^p - 6$ over $\mathbb{Q}$.

(a) Determine the field K and the degree $[K:\mathbb{Q}]$.

(b) Determine all subfields L of K with $[L:\mathbb{Q}] = p-1$.