QUALIFYING EXAM COVER SHEET

August 2017 Qualifying Exams

Instructions: These exams will be "blind-graded" so under the student ID number please use your PUID

ID #: ___________________________
(10 digit PUID)

EXAM (circle one) 519 523 530 544 **553** 554 562 **571**

For grader use:

Points _________ / Max Possible_________ Grade _________

Math 553 **Qualifying Exam** **August, 2017** **W. Heinzer**

Instructions:

1. The point value of each exercise occurs adjacent to the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
11	20	
Total	200	

1. (20) Let $n > 1$ be a positive integer and let p be a prime integer. Let $\varphi : \frac{\mathbb{Z}}{(pn)} \to \frac{\mathbb{Z}}{(n)}$ be the natural surjective ring homomorphism.

(a) If p does not divide n and x is a unit in $\frac{\mathbb{Z}}{(n)}$, is every element in $\varphi^{-1}(x)$ a unit in $\frac{\mathbb{Z}}{(pn)}$? Justify your answer.

(b) If p divides n and x is a unit in $\frac{\mathbb{Z}}{(n)}$, is every element in $\varphi^{-1}(x)$ a unit in $\frac{\mathbb{Z}}{(pn)}$? Justify your answer.

(c) Prove that φ maps the units of $\frac{\mathbb{Z}}{(pn)}$ surjectively onto the units of $\frac{\mathbb{Z}}{(n)}$,

2. (13 pts) Does there exist an infinite abelian group G having the property that every proper subgroup H of G is a finite group? Justify your answer by either describing an example of such a group G, or explaining why such a group G does not exist.

3. (7 pts) Is every nonzero prime ideal of a unique factorization domain a maximal ideal? Justify your answer.

4. (12 pts) Let $\mathbb{Q}$ denote the field of rational numbers. Does there exist a field extension L of $\mathbb{Q}$ such that $[L:\mathbb{Q}] = 4$ and there exist precisely two subfields F_1 and F_2 of L such that $[F_1 : \mathbb{Q}] = 2 = [F_2 : \mathbb{Q}]$? Justify your answer by either describing an example of such a field L, or explaining why such a field L does not exist.

5. (8 pts) Does the symmetric group S_5 contain a subgroup of order 15? Justify your answer.

6. (10) Let R be an integral domain. If $f(x)$ and $g(x)$ are nonzero polynomials in the polynomial ring $R[x]$, prove that their product $f(x)g(x)$ is a nonzero polynomial.

7. (10) Let R be an integral domain and let $f = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ be an element of the formal power series ring $R[[x]]$. State a necessary and sufficient condition for f to be a unit of $R[[x]]$ and justify your answer.

8. (10) Let R be a commutative ring with $1 \neq 0$ and let $f(x)$ and $g(x)$ be polynomials in $R[x]$. Let $c(f)$ and $c(g)$ denote the ideals in R generated by the coefficients of $f(x)$ and $g(x)$, respectively. Assume that $c(f) = R$ and $c(g) = R$. Prove or disprove that $c(fg)$, the ideal in R generated by the coefficients of the product $f(x)g(x)$ is equal to R.

9. (10) How many maximal ideals of the polynomial ring $\mathbb{Z}[[x]$ contain the ideal $(10, x^2 - 3)$? Give generators for each of these maximal ideals.

10. Let G be a finite group and H a subgroup such that $|G : H| = d$ with $1 < d < |G|$.

(a) (5 pts) Describe the natural homomorphism $\phi : G \to S_d$, where S_d is the permutation group on the left cosets of H in G.

(b) (8 pts) If $|G| = n$ and d is the smallest prime dividing n, prove that H is normal in G.

11. (7 pts) Let p and q be distinct prime numbers. List up to isomorphism all abelian groups of order p^3q^2.

12. (6 pts) State Zorn's Lemma.

13. (14 pts) Let R be a commutative ring with $1 \neq 0$. Assume that $a \in R$ is such that $a^n \neq 0$ for each positive integer n, and let $\mathcal{S} = \{a^n\}_{n \geq 0}$.

(i) Using Zorn's Lemma, prove that there exists an ideal I of R such that I is maximal among ideals of R with $I \cap \mathcal{S} = \emptyset$.

(ii) Prove that an ideal I as in item (i) is a prime ideal.

14. Let $K/\mathbb{Q}$ be the splitting field of the polynomial $x^4 + 1 \in \mathbb{Q}[x]$.

(a) (4 pts) What is the degree $[K : \mathbb{Q}]$?

(b) (6 pts) If α is one root of x^4+1, diagram the lattice of fields between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$, and give generators for each intermediate field.

15. Let $L/\mathbb{Q}$ be the splitting field of the polynomial $x^8 - 2 \in \mathbb{Q}[x]$.

(a) (4 pts) What is the degree $[L : \mathbb{Q}]$?

(b) (6 pts) If β is one root of x^8-2, diagram the lattice of fields between $\mathbb{Q}$ and $\mathbb{Q}(\beta)$, and give generators for each intermediate field.

16. (10 pts) Let K/F be an algebraic field extension. If $K = F(\alpha)$ for some $\alpha \in K$, prove that there are only finitely many subfields of K that contain F.

17. (10 pts) Prove or disprove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.

18. (10 pts) Let F be a field of characteristic $p > 0$ and let $F(x)$ denote the field of fractions of the polynomial ring $F[x]$. Let $\operatorname{Aut} F(x)$ denote the group of automorphisms of the field $F(x)$, and let $\sigma \in \operatorname{Aut} F(x)$ be such that σ fixes F and $\sigma x = x + 1$. Let $G = \langle \sigma \rangle$ be the cyclic subgroup of $\operatorname{Aut} F(x)$ generated by σ.

(a) What is the order of the group G?

(b) Give generators for the fixed field $F(x)^G$.

19. (10 pts) Do there exist Galois extensions $K/\mathbb{Q}$ and $L/\mathbb{Q}$ such that $[K : \mathbb{Q}] = 6 = [L : \mathbb{Q}]$, but the Galois groups $\operatorname{Gal}(K/\mathbb{Q})$ and $\operatorname{Gal}(L/\mathbb{Q})$ are not isomorphic? Justify your answer by either giving examples where this holds or explaining why it is not possible.