QUALIFYING EXAM COVER SHEET

August 2021 Qualifying Exams

Instructions: These exams will be "blind-graded" so under the student ID number please use your PUID

ID #: ______________________
(10 digit PUID)

EXAM (circle one) 514 519 523 530 544 **553** 554 562 571

For grader use:

Points ________ / Max Possible________ Grade ________

Qualifying Examination

MA 553

August 10, 2021

Time: 2 hours

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1	
2	
3	
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Total	

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(15 pts) 1). Let p be a prime number, $p \geq 5$, $p \neq 7$, and let n be a positive integer. Show that every group of order $8p^n$ is solvable.

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2).

(10 pts) a) Define the commutator subgroup $[G, G]$ of a group G. Let $H < G$ be a subgroup. Prove that $H > [G, G]$ if and only if H is normal in G and G/H is abelian.

(10 pts) b) Let G be a group and A and B normal subgroups of G with G/A and G/B abelian. Show that $A \cap B$ is normal in G and $G/A \cap B$ is abelian.

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(30 pts) 3). Show that

$$f(x) = (x-1)\dots(x-n) - 1$$

is irreducible over $\mathbb{Z}$ for all integers $n \geq 1$.

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4). Show that the following polynomials are irreducible:

(10 pts) a) $x^3 + 4$ in $\mathbb{Q}[x]$.

(10 pts) b) $x^5 + yx^3 + y^2x^2 + y^n + y$ in $\mathbb{Z}[x, y]$, for a positive integer n.

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5). Let R be the ring

$$R := \mathbb{Z}[x]/(x^3 + x).$$

(15 pts) a) Show that R can be written as a product of euclidean domains. You must verify that they are euclidean domains.

(5 pts) b) Is R a euclidean domain itself? Justify your answer.

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6. Let F be a field of characteristic $p > 0$. Fix an element c in F.

(15 pts) a) Prove that $f(x) = x^p - c$ is irreducible in $F[x]$ if and only if f has no roots in F.

(10 pts) b) Assume $f(x)$ is irreducible in $F[x]$. Let K be a splitting field of $f(x)$ over F. Determine $\mathrm{Aut}(K/F)$, the group of automorphism of K fixing F. Is K/F Galois? Justify your answer.

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7).

(15 pts) a) Show that $\sqrt{5} \notin \mathbb{Q}(\sqrt[3]{2}, \omega)$, where $\omega^2 + \omega + 1 = 0$.

(15 pts) b) Determine the Galois group of

$$f(x) = x^5 - 5x^3 - 2x^2 + 10$$

over $\mathbb{Q}$. (You may assume $\mathrm{Gal}(\mathbb{Q}\sqrt[3]{2}, \omega)/\mathbb{Q}) \simeq S_3$.)

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8).

(20 pts) a) Show that $f(x) = x^4 - 2x^2 - 4$ is irreducible over $\mathbb{Q}$.

(8 pts) b) Show that $\alpha = \sqrt{1+\sqrt{5}}$ is a root of $f(x)$ and let $K = \mathbb{Q}(\alpha)$. Determine a Galois closure L of K over $\mathbb{Q}$.

(6 pts) c) Show that $\sqrt{-1} \in L$.

(6 pts) d) Show that $L/\mathbb{Q}$ is an extension by radicals. Is it solvable? Justify your answer.

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