Qualifying Examination

MA 553

January 12, 2021

Time: 2 hours

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Your ID: ____________________

(15 pts) 1). Show that any group of order 294 is solvable.

Your ID: ____________________________

2).

(5 pts) a) Give the definition of a euclidean domain.

(25 pts) b) Let A be the subring of all the complex numbers $a + b\sqrt{-7}$ in which a and b are both integers or both halves of integers. Prove that A is a euclidean domain. Is A a principal ideal domain (PID)? Prove this. Quoting a theorem is not acceptable.

Your ID: ______________________________

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Your ID: ________________________________

(20 pts) 3). Let α be a root of

$$f(x) = x^{23} - 7x^{15} + 77x^{10} + 35x^6 - 49x^4 + 21 = 0.$$

Is $\mathbb{Q}(\alpha^{13}) = \mathbb{Q}(\alpha)$? Justify your answer.

Your ID: ____________________________________

(25 pts) 4).

(10 pts) a) Let R be a unique factorization domain and let

$$f(x, y) = x^8 + yx^6 + yx^4 + 7yx + y \in R[x, y].$$

Show that $f(x, y)$ is irreducible in $R[x, y]$.

(15 pts) b) Let $K = F(x^8/x^6 + x^4 + 7x + 1)$, where F is a field. Determine $[F(x) : K]$.

Your ID: ______________________________

(30 pts) 5). Show that the irreducible polynomial $x^4 + 1 \in \mathbb{Z}[x]$ is reducible modulo every prime p. (Hint: For odd p show that $x^8 - 1$ divides $x^{p^2-1} - 1$ and thus $x^{p^2} - x$ whose roots are elements of $\mathbb{F}_{p^2}$, finite field with p^2 elements.)

Your ID: ____________________

6.

(15 pts) a) Determine the Galois group of $f(x) = x^3 - 3x + 1$.

(15 pts) b) Show that the Galois group of

$$g(x) = x^5 - 5x^3 + x^2 + 6x - 2$$

is $\mathbb{Z}/6\mathbb{Z}$. The Galois groups are over $\mathbb{Q}$.

Your ID: ________________________________

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Your ID: ______________________________

7). Let $\alpha = \sqrt[3]{1+\sqrt{3}}$ and $\beta = \sqrt[3]{1-\sqrt{3}}$.

(10 pts) a) Show that $[\mathbb{Q}(\alpha)\colon\mathbb{Q}] = 6$.

(10 pts) b) Prove that $K = \mathbb{Q}(\alpha, \beta, \sqrt{-1})$ is a normal closure for $\mathbb{Q}(\alpha)/\mathbb{Q}$.

(10 pts) c) Show that $\sqrt[3]{2} \in K$ and express it as a polynomial in α, β and $\sqrt{-1}$.

(10 pts) d) Show that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{3}, \sqrt{-1})$, but $\omega \in \mathbb{Q}(\sqrt{3}, \sqrt{-1})$, where ω is a root of $\omega^2 + \omega + 1 = 0$. Conclude that $[L\colon\mathbb{Q}] = 12$, where $L = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}, \sqrt{-1})$, and K is obtained by adjoining a cube root of an element in L. Thus $[K\colon\mathbb{Q}] = 12$ or 36.

(10 pts) e) Show that both $K/\mathbb{Q}$ and $L/\mathbb{Q}$ are extensions by radicals. Is $K/\mathbb{Q}$ solvable? Justify your answer.

Your ID: ______________________________

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