QUALIFYING EXAM COVER SHEET

August 2022 Qualifying Exams

Instructions: These exams will be “blind-graded” so under the student ID number please use your PUID

ID #: ____________________________
(10 digit PUID)

EXAM (circle one)  530  544  553  554

For grader use:

Points ________ / Max Possible_________  Grade _________
Qualifying Examination  
MA 553  
August 11, 2022  
Time: 2 hours  
Instructor: F. Shahidi

PUID: ________________________________

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1). (10 pts) a) Show that every solvable group has a non-trivial normal abelian subgroup.

(10 pts) b) Let $G$ be a group and denote by $\text{Aut}(G)$ the group of its automorphisms. Assume $\text{Aut}(G)$ is solvable. Prove that $G$ is solvable.
2). Let $p$ and $q$ be two prime numbers with $p < q$. Let $G$ be a group of order $pq$.

(a) Assume $p$ does not divide $q - 1$. Show that $G$ is cyclic which is in fact a direct product of a $q$-Sylow ($S_q$) subgroup $Q$ and a $p$-Sylow ($S_p$) subgroup $P$ of $G$.

(b) Assume $p | q - 1$ and $G$ is not cyclic. Conclude that in this case $G$ is non-abelian and is a semi-direct product of a $S_q$-subgroup $Q$ and a $S_p$-subgroup $P$ of $G$, but not their direct product.

(c) Let $p$ and $q$ be two primes as above with $p | q - 1$. Let $P$ and $Q$ be the (cyclic) groups of orders $p$ and $q$, respectively. Show that all the semi-direct products $Q \rtimes_{\varphi} P$, where $\varphi : P \rightarrow \text{Aut}(Q)$ — are non-trivial homomorphisms, are isomorphic. You may assume the fact that finite subgroups of the multiplicative group of a field are cyclic.
(20 pts) 3). Let $\alpha$ be a root of

$$f(x) = x^{23} - 5x^{19} + 25x^{11} - 30x^8 + 35x^5 + 10 = 0$$

Is $\mathbb{Q}(\alpha^{10}) = \mathbb{Q}(\alpha)$? Justify your answer.
4). 

(a) Let $R$ be a commutative ring. Let $I$ and $J$ be two ideals in $R$. Assume $P$ is a prime ideal of $R$ such that $I \cap J \subset P$. Show that either $I$ or $J$ is contained in $P$.

(b) Show that $f(x, y) = x^2 + xy + y^2 + y$ is irreducible in $\mathbb{Z}[x, y]$.
5).

(10 pts)  a) Show that polynomial $f(x) = x^4 + 1$ is irreducible in $\mathbb{Z}[x]$ using Eisenstein Criterion.

(20 pts)  b) Show that $f(x) = x^4 + 1$ is reducible modulo every prime $p$. (Hint: For odd $p$ show that $x^8 - 1$ divides $x^{p^2} - x$ whose roots are elements of a field with $p^2$ elements.)

Thus a polynomial in $\mathbb{Z}[x]$ could be irreducible over $\mathbb{Z}$, but reducible over every $\mathbb{Z}/p\mathbb{Z}$, $p$ a prime number.
6).

(5 pts)  a) Define the discriminant of a polynomial of degree $n$ over $\mathbb{Q}$.

(20 pts) b) Use Galois theory to prove that a cubic polynomial over $\mathbb{Q}$, not necessarily irreducible, has only real roots iff its discriminant is non-negative.
7). (8 pts) a) Using that \( \mathbb{Z}[^{-1}\sqrt{1}] \) is a unique factorization domain (UFD) show that

\[ x^3 - (1 + \sqrt{-1}) \]

is irreducible over \( \mathbb{Z}[\sqrt{-1}] \) and \( \mathbb{Q}(\sqrt{-1}) \).

(7 pts) b) Show that the polynomial \( f(x) := x^6 - 2x^3 + 2 \) is irreducible over \( \mathbb{Q} \) which has \( \alpha = 3\sqrt{1 + \sqrt{-1}} \) and \( \beta = 3\sqrt{1 - \sqrt{-1}} \) among its roots. What is \([\mathbb{Q}(\alpha) : \mathbb{Q}]\)?

(10 pts) c) Determine the irreducible polynomial for a primitive 12\(^{th}\) root of unity (12\(^{th}\) cyclotomic polynomial).

(10 pts) d) Let \( L = \mathbb{Q}(\alpha, \beta) \). Show that \( 3\sqrt{2} \in L \).

Using part c) prove that \( 3\sqrt{2} \notin \mathbb{Q}(\alpha) \).

What is \([L : \mathbb{Q}]\)?

(5 pts) e) Show that \( K = \mathbb{Q}(\alpha, \sqrt{2}, \sqrt{3}) \) is a splitting field for \( f(x) \) over \( \mathbb{Q} \).

(5 pts) f) Show that \( K/\mathbb{Q} \) is an extension by radicals.

Is it solvable? Justify your answer.