QUALIFYING EXAM COVER SHEET

August 2022 Qualifying Exams

Instructions: These exams will be "blind-graded" so under the student ID number please use your PUID

ID #: ____________________________
(10 digit PUID)

EXAM (circle one) 530 544 **553** 554

For grader use:

Points _________ **/ Max Possible**_________ **Grade** _________

Qualifying Examination
MA 553
August 11, 2022
Time: 2 hours
Instructor: F. Shahidi

PUID: ______________________

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MA 553 Qualifying Examination August 11, 2022 Inst: F. Shahidi

Your ID: ______________________________

1).

(10 pts) a) Show that every solvable group has a non-trivial normal abelian subgroup.

(10 pts) b) Let G be a group and denote by $\mathrm{Aut}(G)$ the group of its automorphisms. Assume $\mathrm{Aut}(G)$ is solvable. Prove that G is solvable.

Your ID: ______________________________

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MA 553 Qualifying Examination August 11, 2022 Inst: F. Shahidi

Your ID: ______________________

2). Let p and q be two prime numbers with $p < q$. Let G be a group of order pq.

(10 pts) a) Assume p does not divide $q-1$. Show that G is cyclic which is in fact a direct product of a q-Sylow (S_q) subgroup Q and a p-Sylow (S_p) subgroup P of G.

(15 pts) b) Assume $p|q-1$ and G is not cyclic. Conclude that in this case G is non-abelian and is a semi-direct product of a S_q-subgroup Q and a S_p-subgroup P of G, but not their direct product.

(15 pts) c) Let p and q be two primes as above with $p|q-1$. Let P and Q be the (cyclic) groups of orders p and q, respectively. Show that all the semi-direct products $Q \rtimes_\varphi P$, where $\varphi : P \to \mathrm{Aut}(Q)$ – are non-trivial homomorphisms, are isomorphic. You may assume the fact that finite subgroups of the multiplicative group of a field are cyclic.

Your ID: ______________________________

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Your ID: ____________________

(20 pts) 3). Let α be a root of

$$f(x) = x^{23} - 5x^{19} + 25x^{11} - 30x^8 + 35x^5 + 10 = 0$$

Is $\mathbb{Q}(\alpha^{10}) = \mathbb{Q}(\alpha)$? Justify your answer.

Your ID: ______________________

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Your ID: ____________________

4).

(10 pts) a) Let R be a commutative ring. Let I and J be two ideals in R. Assume P is a prime ideal of R such that $I \cap J \subset P$. Show that either I or J is contained in P.

(10 pts) b) Show that $f(x, y) = x^2 + xy + y^2 + y$ is irreducible in $\mathbb{Z}[x, y]$.

Your ID: ______________________________

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Your ID: ______________________________

5).

(10 pts) a) Show that polynomial $f(x) = x^4 + 1$ is irreducible in $\mathbb{Z}[x]$ using Eisenstein Criterion.

(20 pts) b) Show that $f(x) = x^4 + 1$ is reducible modulo every prime p. (Hint: For odd p show that $x^8 - 1$ divides $x^{p^2} - x$ whose roots are elements of a field with p^2 elements.)

Thus a polynomial in $\mathbb{Z}[x]$ could be irreducible over $\mathbb{Z}$, but reducible over every $\mathbb{Z}/p\mathbb{Z}$, p a prime number.

Your ID: ________________________________

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Your ID: ______________________

6).

(5 pts) a) Define the discriminant of a polynomial of degree n over $\mathbb{Q}$.

(20 pts) b) Use Galois theory to prove that a cubic polynomial over $\mathbb{Q}$, not necessarily irreducible, has only real roots iff its discriminant is non-negative.

Your ID: ______________________________

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Your ID: ______________________________

7).

(8 pts) a) Using that $\mathbb{Z}[\sqrt{-1}]$ is a unique factorization domain (UFD) show that

$$x^3 - (1 + \sqrt{-1})$$

is irreducible over $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Q}(\sqrt{-1})$.

(7 pts) b) Show that the polynomial $f(x) := x^6 - 2x^3 + 2$ is irreducible over $\mathbb{Q}$ which has $\alpha = \sqrt[3]{1 + \sqrt{-1}}$ and $\beta = \sqrt[3]{1 - \sqrt{-1}}$ among its roots. What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$?

(10 pts) c) Determine the irreducible polynomial for a primitive 12^{th} root of unity (12^{th} cyclotomic polynomial).

(10 pts) d) Let $L = \mathbb{Q}(\alpha, \beta)$. Show that $\sqrt[3]{2} \in L$.

Using part c) prove that $\sqrt[3]{2} \notin \mathbb{Q}(\alpha)$.

What is $[L : \mathbb{Q}]$?

(5 pts) e) Show that $K = \mathbb{Q}(\alpha, \sqrt[3]{2}, \sqrt{3})$ is a splitting field for $f(x)$ over $\mathbb{Q}$.

(5 pts) f) Show that $K/\mathbb{Q}$ is an extension by radicals.

Is it solvable? Justify your answer.

Your ID: ______________________________

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