Qualifying Examination

MA 553

January 4, 2022

Time: 2 hours

Your ID: ______________________________

1	
2	
3	
4	
5	
6	
7	
Total	

Your ID: ______________________________

1).

(15 pts) a) Let G be a group and $\text{Aut}(G)$ its group of automorphism. Assume $\text{Aut}(G)$ is abelian. Show that G is solvable.

(10 pts) b) Is the converse true? Justify your answer.

Your ID: ______________________________

2). Let $R = \mathbb{Z}[\sqrt{-5}\,]$ and let $I = (2, 1 + \sqrt{-5})$, $J = (3, 2 + \sqrt{-5})$, and $K = (3, 2 - \sqrt{-5})$ be ideals in R.

(10 pts) a) Show that none of these three ideals are principal.

(25 pts) b) Show that $IJ = (1 - \sqrt{-5})$, $IK = (1 + \sqrt{-5})$, and $I^2JK = (6)$.

(5 pts) c) Show that $I^2 = (2)$.

Thus all these products are principal.

Your ID: ________________________________

(15 pts) 3). Let α be a root of

$$f(x) = x^{19} - 5x^{16} + 25x^{12} + 35x^7 - 50x^3 + 15 = 0.$$

Is $\mathbb{Q}(\alpha^{13}) = \mathbb{Q}(\alpha)$? Justify your answer.

Your ID: ______________________________

4).

(10 pts) a) Show that the polynomial $f(x) = x^3 - 3x + 1$ is irreducible over $\mathbb{Z}$ and calculate its discriminant to conclude that it has three (distinct) real roots. What is its Galois group?

(10 pts) b) Show that $x - 1$ and $x^3 - 3x + 1$ are relatively prime in $\mathbb{Z}[x]$, i.e., the ideal generated by them is $\mathbb{Z}[x]$.

(10 pts) c) Give a simpler description of the ring

$$\mathbb{Z}[x]/((x-1)(x^3-3x+1)).$$

Your ID: ______________________________

(25 pts) 5). Prove that one of $2, 3$ or 6 is a square in any field $\mathbb{Z}/p\mathbb{Z}$, where p is a prime. Conclude that the polynomial

$$x^6 - 11x^4 + 36x^2 - 36 = (x^2 - 2)(x^2 - 3)(x^2 - 6)$$

has a root modulo p for every prime p but has no root in $\mathbb{Z}$.

Your ID: ______________________________

6).

(10 pts) a) Define an extension by radicals.

(20 pts) b) Let E be a finite Galois extension of $\mathbb{Q}$ of degree pq^m, where p and q are prime numbers with $p < q$ and m is a non-negative integer. Prove that every irreducible polynomial over $\mathbb{Q}$ which splits in E is solvable by radicals.

(5 pts) c) Let $E/\mathbb{Q}$ be a Galois extension of degree 4802. Show that every irreducible polynomial over $\mathbb{Q}$ which splits in E is solvable by radicals.

Your ID: ______________________________

7).

(10 pts) a) Let ξ_3 and ξ_6 be a third and a sixth primitive root of 1, respectively, such that $\xi_6^2 = \xi_3$. Show that $\mathbb{Q}(\xi_3) = \mathbb{Q}(\xi_6)$, and write $Irr(\mathbb{Q}(\xi_3), \xi_6)$ as a linear equation with coefficients in $\mathbb{Q}(\xi_3)$.

(10 pts) b) Let

$$f(x) = (x^3 - 2)(x^2 - x + 1).$$

Give a splitting field K of $f(x)$ over $\mathbb{Q}$ and determine its Galois group $Gal(K/\mathbb{Q})$.

(10 pts) c) Determine all the subfields L of K, $K \supset L \supset \mathbb{Q}$.