Math 553 **Name:** ______________________

January 2023 **Qualifying Exam**

1/5/23 **10 am-12 pm**

- **DO NOT open the exam booklet until you are told to begin. You should write your name and read the instructions.**
- Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.
- You may use any results from class, homework, or Dummit and Foote that does not trivialize the problem, but you must cite the result you are using.
- You may use a fact without proof if its verification is straightforward and you explicitly say so.
- You needn't spend your time rewriting definitions or axioms on the exam.
- You may use any textbooks, my class notes, or any notes and study guides you have created. You may not use a cell phone, computer, or any other electronics.
- When you have completed your test, hand it to me.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	20	
Total:	90	

1. (10 points) Show that any group of order 15 is cyclic i.e. is isomorphic to $\mathbb{Z}/15$.

2. (10 points) Let p be a prime, n be a positive integer, and G be a p-group. Show that a G-action on $\mathbb{F}_p^n$ by linear transformations (i.e. $g \cdot (c\vec{v} + \vec{w}) = c(g \cdot \vec{v}) + g \cdot \vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{F}_p^n$ and $c \in \mathbb{F}_p$) has a nonzero fixed vector (i.e. a nonzero vector with stabilizer equal to G).

3. (10 points) Find (with proof) a finite set of generators for the ideal

$$x\mathbb{Q}[x, y] \cap y\mathbb{Q}[x, y].$$

4. (10 points) Let k be a field. Show that $k[x, y]/(y^2 - x^3 + x^2)$ is an integral domain.

5. (10 points) Show that $x^4 + 1 \in \mathbb{Q}[x]$ is irreducible. *Hint: Let* $y = x - 1$.

6. (10 points) Let $\zeta_8 = e^{2\pi i/8} \in \mathbb{C}$. Classify, with proof, all quadratic extensions of $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta_8)$. For each quadratic extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta_8)$, write a generator of this extension over $\mathbb{Q}$ as a polynomial in ζ_8.

7. (10 points) Let K/F be the splitting field of an irreducible and separable quintic (degree 5) polynomial $f(x) \in F[x]$. Show that if $K = F(\alpha, \beta)$ for two roots α, β of $f(x)$, then $[K : F] = 5, 10$, or 20. *Hint: Partial credit for showing that* $5 \mid [K : F]$ *and* $[K : F] \leq 20$.

8. Let K/F be the splitting field of a separable quintic (degree 5) polynomial $f(x) \in F[x]$. Suppose that $\mathrm{Gal}(K/F) \cong S_5$. Let $\alpha, \beta, \gamma, \delta, \varepsilon \in K$ be the distinct roots of $f(x)$.

   (a) (10 points) Show that $F(\alpha, \beta)$ and $F(\gamma, \delta)$ are isomorphic fields.

(b) (10 points) Determine (with proof) $\mathrm{Gal}(K/F(\alpha, \beta))$ and $\mathrm{Gal}(K/F(\alpha + \beta, \alpha\beta))$.