QUALIFYING EXAMINATION

August 2023

MA 553

Problem 1. (17 points) Let $n \geq 4$ be an integer. Prove that every group of order $10 \cdot 3^n$ has a normal subgroup of order 3^k for some $k \geq n-4$.

Problem 2. (21 points) Prove that up to isomorphisms there are at most four groups of order 66. (Hint: Show that a group of order 66 has a cyclic subgroup of order 33.)

Problem 3. (7 points) Let G be a solvable group. Show that G has a composition series if and only if G is finite.

Problem 4. (10 points) Show that if R is a unique factorization domain whose quotient field is isomorphic to $\mathbb{R}$ then $R \cong \mathbb{R}$. (Hint: If R is not a field, then R has a prime element.)

Problem 5. (18 points) Let $\alpha \in \mathbb{C}$ be the root of a polynomial in $\mathbb{Z}[x]$ that is monic (i.e., has leading coefficient 1). Let $\mathbb{Z}[\alpha]$ be the smallest subring of $\mathbb{C}$ containing α, let $q \in \mathbb{Q}[x]$ be the minimal polynomial of α over $\mathbb{Q}$, and write $d = \deg q$. Show that

(a) $q \in \mathbb{Z}[x]$;

(b) $\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(q)$;

(c) as an additive Abelian group, $\mathbb{Z}[\alpha]$ is generated by $\{\alpha^i \mid 0 \leq i \leq d-1\}$.

Problem 6. (13 points) Let $k \subset K$ be a field extension that is Galois with Galois group G and let L_1, L_2 be intermediate fields. Prove that L_1 and L_2 are isomorphic over k if and only if the subgroups $\mathrm{G}(K/L_1)$ and $\mathrm{G}(K/L_2)$ of G are conjugate in G.

Problem 7. (14 points) Let $f = x^3 - 3x + 3 \in \mathbb{Q}[x]$ and let K be the splitting field of f over $\mathbb{Q}$. Determine

(a) the Galois group $\mathrm{G}(K/\mathbb{Q})$ (up to isomorphism);

(b) the number of distinct subfields of K.