# 553 QUALIFYING EXAM. FALL 2025

Attempt all questions. Time 2 hrs.

1. (10 pts) Let $G$ be a finite group in which $x^2 = e$ for all elements $x \in G$. Prove that the order of $G$ is a power of 2.
2. (10 pts) Let $p$ be a prime integer and $G$ be a $p$-group. If $H$ is a normal subgroup of $G$ and $|H| = p$, prove that $H$ is contained in the center of $G$.
3. (10 pts) Let $p < q$ be prime numbers such that $p$ divides $q - 1$. Show that there exists a non-abelian group of order $pq$.
4. (10 pts) Let $G$ be the subgroup of $GL_2(\mathbb{R})$ consisting of all matrices of the form
$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$
Show that $G$ is a solvable group.
5. (10 pts) If $A$ is an integral domain which is not a field, prove that $A[x]$ is not a principal ideal domain.
6. (2.5 + 2.5 + 2.5 + 2.5 pts) In which of the following rings is every ideal principal? Justify your answer in each case.
$$(i)\ \mathbb{Z} \times \mathbb{Z}, \qquad (ii)\ \frac{\mathbb{Z}}{(4)}, \qquad (iii)\ \frac{\mathbb{Z}}{(6)}[x], \qquad (iv)\ \frac{\mathbb{Z}}{(4)}[x].$$
7. (5 + 5 pts) Let $R$ be an integral domain.
   (a) Define irreducible elements and prime elements of $R$.
   (b) Prove that every prime element of $R$ is an irreducible element.
8. (5 + 5 pts) Let $p$ be a prime number and $\phi_p = X^{p-1} + \cdots + 1 \in \mathbb{Q}[X]$. Prove that $K = \mathbb{Q}[X]/(\phi_p)$ is a splitting field of $\phi_p$ and $K/\mathbb{Q}$ is a Galois extension. What is the Galois group of the extension $K/\mathbb{Q}$ ?
9. (2+4+4 pts)
   (a) Define Galois extensions of fields.
   (b) State but do NOT prove the Galois correspondence theorem.
   (c) Describe the Galois correspondence for the extension of $\mathbb{Q}$ by the splitting field of the polynomial $X^4 - 2$.
10. (2+4+4 pts)
   (a) What is a separable field extension ?
   (b) Prove that every irreducible polynomial in $k[X]$ is separable if char $k = 0$.
   (c) Prove that every irreducible polynomial in $k[X]$ is separable if $k$ is a finite field.