Math 553 **Name:** ______________________

January 2025 **Qualifying Exam**

1/9/25 **10 am-12 pm**

- **DO NOT open the exam booklet until you are told to begin. You should write your name and read the instructions.**
- Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.
- You may use any results from class, homework, or Dummit and Foote that do not trivialize the problem, but you must cite the result you are using.
- You may use a fact without proof if its verification is straightforward and you explicitly say so.
- You needn't spend your time rewriting definitions or axioms on the exam.
- You may not use any aids including any electronics.
- When you have completed your test, hand it to me.

Problem	Points	Score
1	20	
2	25	
3	10	
4	30	
5	35	
Total:	120	

1. Let G be the permutation group S_5 acting on the set $\{1, 2, 3, 4, 5\}$. Let H be the stabilizer of the element 5.

   (a) (10 points) The group $H \times H$ acts on G by left and right multiplication: $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. Determine the number of orbits for this action. Prove your answer.

(b) (10 points) Find a subgroup $K \subset G$ such that G is a semidirect product of K and H, or show that that no such subgroup exists.

2. (10 points) (a) (5 points) Find generators for the normalizer in S_6 of the subgroup H generated by $(12), (34)$, and (56).

(b) (5 points) Show that any 2-Sylow subgroup of S_6 containing H is contained in the normalizer of H.

(c) (5 points) Find generators for a Sylow-2-subgroup of S_6 containing H in 2(a).

3. (10 points) Let k be a field and $k(t)$ be the field of rational functions in one variable t over k. Find an element in $k(t)$ whose square is $t \in k(t)$, or show that no such element exists.

4. (10 points) Let p be a prime number.

   (a) (10 points) Show that $x^p - x - a \in \mathbb{F}_p[x]$ is irreducible for all $a \in \mathbb{F}_p^\times$. *Hint: Look at the action of the Frobenius automorphism.*

(b) (10 points) Show that the splitting fields of $x^p - x - a \in \mathbb{F}_p[x]$ for $a \in \mathbb{F}_p^\times$ are all isomorphic.

5. Let $\alpha = \sqrt{1+\sqrt{3}} \in \mathbb{R}$ where for a positive real number r, $\sqrt{r} \in \mathbb{R}$ denotes the positive square root.

   (a) (5 points) Find, with proof, the minimal (monic degree 4) polynomial $f(x) \in \mathbb{Q}[x]$ of α over $\mathbb{Q}$. *Make sure to explain why your answer is irreducible!*

(b) (10 points) Let K be a splitting field over $\mathbb{Q}(\alpha)$ of $f(x)$ from 5(a). Show that $[K : \mathbb{Q}] = 8$.

(c) (10 points) The group $G = \mathrm{Gal}(K/\mathbb{Q})$ acts faithfully on the roots of the polynomial $f(x)$ from 5(a). Order the 4 roots in the sequence $\alpha, -\alpha, \beta_1, \beta_2$ to identify G with a subgroup of S_4, and find generators in S_4 for this subgroup.

(d) (10 points) Show that there are exactly three subfields of K which are quadratic extensions of $\mathbb{Q}$. Find generators for these subfields over $\mathbb{Q}$.