PUID: ________________________________

Instructions:
1. The point value of each exercise occurs to the left of the problem.
2. No books or notes or calculators are allowed.

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1. (20 pts) Let $V$ be an abelian group generated by elements $a, b, c$. Assume the following relations hold: $2a = 4b, 2b = 4c, 2c = 4a$, and these three relations generate all the relations on $a, b, c$.

(a) Write down a relation matrix for $V$.

(b) Find generators $x, y, z$ for $V$ such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by $x, y, z$, and express your generators $x, y, z$ in terms of $a, b, c$.

(c) What is the order of $V$?

(d) What is the order of the element $a$?
2. (21 pts) Let \( T : V \to V \) be a linear operator on an \( n \)-dimensional vector space over a field \( F \). Let \( c_1, \ldots, c_k \) be distinct elements in \( F \) and let \( p(x) = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k} \) be the minimal polynomial of \( T \). Let \( W_i = \{ v \in V \mid (T - c_i I)^{r_i}(v) = 0 \} \).

(a) Describe linear operators \( E_i : V \to V \), \( i = 1, \ldots, k \), such that \( E_i(V) = W_i \), \( E_i^2 = E_i \) for each \( i \), \( E_i E_j = 0 \) if \( i \neq j \), and \( E_1 + \cdots + E_k = I \) is the identity operator on \( V \).

(b) Describe how to obtain linear operators \( D \) and \( N \) such that \( T = D + N \), where \( D \) is diagonalizable, \( N \) is nilpotent and \( D \) and \( N \) are polynomials in \( T \).

(c) If \( T = D' + N' \), where \( D' \) is diagonalizable and \( N' \) is nilpotent and \( D' N' = N' D' \), prove that \( D = D' \) and \( N = N' \).
3. (21 pts) Let notation be as in the previous problem and let \( f(x) = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k} \) be the characteristic polynomial for \( T \). Thus \( n = d_1 + \cdots + d_k \) and \( 1 \leq r_i \leq d_i \) for each \( i \).

(a) If \( r_i + 1 = d_i \) for each \( i \in \{1, \ldots, k\} \), describe the Jordan form for \( T \).

(b) If \( r_i + 2 = d_i \) for each \( i \in \{1, \ldots, k\} \), how many different Jordan forms are possible for \( T \)?

(c) If \( r_i + 3 = d_i \) for each \( i \in \{1, \ldots, k\} \), how many different Jordan forms are possible for \( T \)?
4. (10 pts) Let $V$ be a finite-dimensional vector space over an infinite field $F$. Prove that $V$ is not the union of finitely many proper subspaces.

5. (10 pts) Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $\alpha_1, \ldots, \alpha_m$ be finitely many nonzero vectors in $V$. Prove that there exists a linear functional $f$ on $V$ such that $f(\alpha_i) \neq 0$ for each $i$ with $1 \leq i \leq m$. 
6. (20 pts) Let $V$ be a finite dimensional inner product space over $\mathbb{C}$ and let $T : V \to V$ be a linear operator.

(a) (2 pts) Define the adjoint $T^*$ of $T$.

(b) (6 pts) If $T = T^*$, prove that every characteristic value of $T$ is a real number.

(c) (6 pts) Assume that $T = T^*$ and that $c$ and $d$ are distinct characteristic values of $T$. If $\alpha$ and $\beta$ in $V$ are such that $T\alpha = c\alpha$ and $T\beta = d\beta$, prove that $\alpha$ and $\beta$ are orthogonal.

(d) (6 pts) State true or false and justify: If $A \in \mathbb{R}^{5 \times 5}$ is symmetric, then $A$ is diagonalizable.
7. Let $M$ be a module over the integral domain $D$. A submodule $N$ of $M$ is pure in $M$ if the following holds: given $y \in N$ and $a \in D$ such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(a) (10 pts) Let $N$ be a submodule of $M$ and for $x \in M$, let $x + N$ denote the coset representing the image of $x$ in the quotient module $M/N$. If $N$ is a pure submodule of $M$, and $\text{ann} \ x = \{ a \in D \mid ax = 0 \}$ is the principal ideal $(d)$ of $D$, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\text{ann} \ x' = \{ a \in D \mid ax' = 0 \}$ is the principal ideal $(d)$.

(b) (10 pts) If $M = \langle \alpha \rangle$ is a cyclic $\mathbb{Z}$-module of order 12, list the submodules of $M$ and indicate which of the submodules of $M$ are pure in $M$. 
8. (16 pts) Let $M$ be a finitely generated module over the polynomial ring $F[x]$, where $F$ is a field, and let $N$ be a pure submodule of $M$. Prove that there exists a submodule $L$ of $M$ such that $N + L = M$ and $N \cap L = 0$. 
9. (12 pts) Prove or disprove: if $V$ is a vector space over a field $F$ and $T : V \to V$ is a linear operator such that every subspace of $V$ is invariant under $T$, then $T$ is a scalar multiple of the identity operator.

10. Let $F$ be a field and let $g(x) \in F[x]$ be a monic polynomial.

   (a) (5 pts) Describe the $F[x]$-submodules of $V = F[x]/(g(x))$.

   (b) (5 pts) If $g(x) = x^3(x - 1)$, diagram the lattice of $F[x]$-submodules of $V = F[x]/(g(x))$. 
11. (16 pts) Classify up to similarity all $3 \times 3$ complex matrices $A$ such that $A^3 = I$, the identity matrix. How many equivalence classes are there?
12. (8 pts) Let $V$ be an abelian group with generators $(v_1, v_2, v_3)$ that has the matrix
\[
\begin{bmatrix}
2 & 0 & 6 \\
6 & 12 & 0
\end{bmatrix}
\]
as a relation matrix. Express $V$ as a direct sum of cyclic groups.

13. (16 pts) Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(a) Write down a relation matrix for $V$ as a $\mathbb{Z}$-module.

(b) Let $W$ be the cyclic subgroup of $V$ generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$. Write down a relation matrix for $W$.

(c) Write down a relation matrix for the quotient module $V/W$.

(d) What is the cardinality of the quotient module $V/W$?