Math 554 **Qualifying Exam** **August 1995**

Name: ____

(10) 1. Let V be an abelian group and assume that $(v_1, \ldots, v_m)$ are generators of V. Describe a process for obtaining an $m \times n$ matrix $A \in \mathbb{Z}^{m \times n}$ such that if $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$ is the $\mathbb{Z}$-module homomorphism defined by left multiplication by A, then $V \cong \mathbb{Z}^m / \phi(\mathbb{Z}^n)$. Such a matrix A is called a presentation matrix of V.

(15) 2. Consider the abelian group $V = \mathbb{Z}/(5^3) \oplus \mathbb{Z}/(5^2) \oplus \mathbb{Z}/(5^2)$.

(1) Write down a presentation matrix for V as a $\mathbb{Z}$-module.

(2) Let W be the cyclic subgroup of V generated by the image of $(10, 2, 1)$ in $\mathbb{Z}/(5^3) \oplus \mathbb{Z}/(5^2) \oplus \mathbb{Z}/(5^2) = V$. Write down a presentation matrix for W.

(3) Write down a presentation matrix for the quotient $\mathbb{Z}$-module V/W.

(20) 3. Let R be a commutative ring and let V and W denote free R-modules of rank 4 and 5, respectively. Assume that $\phi : V \to W$ is an R-module homomorphism, and that $\mathbf{B} = (v_1, \ldots, v_4)$ is an ordered basis of V and $\mathbf{B}' = (w_1, \ldots, w_5)$ is an ordered basis of W.

(1) What is meant by the coordinate vector of $v \in V$ with respect to the basis $\mathbf{B}$?

(2) Describe how to obtain a matrix $A \in R^{5\times 4}$ so that left multiplication by A on R^4 represents $\phi : V \to W$ with respect to $\mathbf{B}$ and $\mathbf{B}'$.

(3) How does the matrix A change if we change the basis $\mathbf{B}$ by replacing v_1 by $v_1 + v_2$?

(4) How does the matrix A change if we change the basis $\mathbf{B}'$ by replacing w_1 by $w_1 + w_2$?

(18) 4. Let A be an 4×5 matrix with coefficients in a commutative ring R and let $\phi : R^5 \to R^4$ be defined by left multipliation by A.

(1) Prove or disprove: if ϕ is surjective, then the determinants of the 4×4 minors of A generate the unit ideal of R.

(2) Prove or disprove: if ϕ is surjective, then there exists a matrix $B \in R^{5 \times 4}$ such that AB is the 4×4 identity matrix.

(10) 5. Let $V = \mathbb{Z}^2$ and let L be the submodule of V spanned by the columns of $A = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$. Find a basis $(\vec{\alpha}_1, \vec{\alpha}_2)$ of V and integers c_1, c_2 so that $c_1\vec{\alpha}_1,\ c_2\vec{\alpha}_2$ is a basis for L.

(10) 6. Let K be the $\mathbb{Z}$-submodule of $\mathbb{Z}^3$ generated by

$$f_1 = (1, 0, 4), \qquad f_2 = (1, -2, 2), \qquad f_3 = (2, 2, -4).$$

Prove or disprove that there exists an integer n and a $\mathbb{Z}$-module homomorphism $\phi : \mathbb{Z}^3 \to \mathbb{Z}^n$ such that $\ker \phi = K$.

(16) 7. Let F be a field and let $R = F[t]$ be a polynomial ring in one variable over F. Let r and s and $a_1 \geq a_2 \geq \cdots \geq a_r$ and $b_1 \geq b_2 \geq \cdots \geq b_s$ be positive integers. Suppose

$$V = R/(t^{a_1}) \oplus R/(t^{a_2}) \oplus \cdots \oplus R/(t^{a_r})$$

and

$$W = R/(t^{b_1}) \oplus R/(t^{b_2}) \oplus \cdots \oplus R/(t^{b_s}).$$

If the R-modules V and W are isomorphic, prove the structure theorem that asserts that $r = s$, and that $a_i = b_i$ for $i = 1, \ldots, r$.

(14) 8. Over the ring $\mathbb{Z}[i]$ of Gaussian integers, let V be the $\mathbb{Z}[i]$-module generated by the two elements v_1, v_2 with relations $(1+i)v_1 + 2v_2 = 0$ and $4v_1 + (1+i)v_2 = 0$. Write V as a direct sum of cyclic $\mathbb{Z}[i]$-modules.

(8) 9. Determine the number of isomorphism classes of abelian groups of order 200. Justify your answer.

(15) 10. Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear operator.

(1) If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, prove that $\operatorname{im}(T) \cap \ker(T) = 0$.

(2) If $\dim(V) = n$, prove that $\operatorname{rank}(T^n) = \operatorname{rank}(T^{n+1})$.

(3) If $\dim(V) = n$, prove that $V = \operatorname{im}(T^n) \oplus \ker(T^n)$.

(18) 11. Let F be a field and let $F[t]$ be a polynomial ring in one variable over F. Let $p(t) = t^n + a_{n-1} + \cdots + a_1 t + a_0 \in F[t]$ be a monic polynomial.

(1) Write down a matrix $A \in F^{n\times n}$ having characteristic polynomial $p(t)$.

(2) Prove the Cayley-Hamilton Theorem that if $p(t) \in F[t]$ is the characteristic polynomial of a matrix $B \in F^{n\times n}$, then $p(B) = 0$.

(8) 12. Let R be a commutative ring, let V be an R-module, and let W be a submodule of V. If W and V/W are finitely generated R-modules, prove that V is a finitely generated R-module.

(8) 13. Let $\mathbb{F}_7$ denote the prime field with 7 elements. What is the order of the group $\mathrm{GF}_3(\mathbb{F}_7)$ of 3×3 invertible matrices with entries in $\mathbb{F}_7$? Justify your answer.

(18) 14. Let T be a linear operator on $\mathbb{C}^2$ defined by the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ with respect to some basis of $\mathbb{C}^2$. Let V denote the module over the polynomial ring $\mathbb{C}[t] = R$ associated to T. Recall that an R-module is said to be *indecomposable* if it is not the direct sum of two nonzero submodules.

(1) Prove or disprove that V is an indecomposable R-module.

(2) Prove or disprove that V is a cyclic R-module.

(12) 15. Let $P \in \mathbb{R}^{5\times 5}$ be such that $P^2 = P^T$, where P^T denotes the transpose of P. Regarding $P \in \mathbb{C}^{5\times 5}$ what are the possible eigenvalues of P? Justify your answer.