Math 554 **Qualifying Exam** **January 1995**

Name: ______________________________

(16) 1. Let S and T be linear operators on a vector space V.

(i) If V is finite-dimensional and the composite $S \cdot T = I$, the identity operator on V, prove that $T \cdot S = I$.

(ii) Prove or disprove that the assertion in part (i) is still true when V is infinite-dimensional.

(16) 2. (i) Let V be a vector space and let U and W be subspaces of V. Prove that the quotient spaces $(U + W)/W$ and $U/(U \cap W)$ are isomorphic.

(ii) Let V be a module over a commutative ring and let U and W be submodules of V. Prove or disprove that the quotient modules $(U+W)/W$ and $U/(U\cap W)$ are isomorphic.

(16) 3. (i) Suppose $A, B \in \mathbb{C}^{3\times 3}$ have the same characteristic polynomial and the same minimal polynomial. Prove or disprove that A and B are similar.

(ii) Suppose $A, B \in \mathbb{C}^{4\times 4}$ have the same characteristic polynomial and the same minimal polynomial. Prove or disprove that A and B are similar.

(16) 4. Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ be a $\mathbb{Z}$-module homomorphism given by multiplication by $A \in \mathbb{Z}^{n\times n}$.

(i) Prove that the image of φ is of finite index if and only if $\det(A) \neq 0$.

(ii) Assuming the image of φ is of finite index, how does this index relate to $\det(A)$? Justify your answer.

(14) 5. Let V be the $\mathbb{Z}[i]$-module generated by elements v_1, v_2 with relations $(1+i)v_1 + (2-i)v_2 = 0$ and $3v_1 + 5iv_2 = 0$. Write V as a direct sum of cyclic $\mathbb{Z}[i]$-modules.

(16) 6. Let $V = \mathbb{Z}^2$ and let L be the submodule of V spanned by the columns of $A = \begin{bmatrix} 6 & -2 \\ 2 & 4 \end{bmatrix}$. Find a basis $(\vec{\alpha}_1, \vec{\alpha}_2)$ of V and integers c_1, c_2 so that $c_1\vec{\alpha}_1,\ c_2\vec{\alpha}_2$ is a basis for L.

(16) 7. Let $P \in \mathbb{R}^{5\times 5}$ be such that $P^2 = P^t$, where P^t denotes the transpose of P. Regarding $P \in \mathbb{C}^{5\times 5}$ what are the possible eigenvalues of P? Justify your answer.

(18) 8. Let F be a field and let n be a positive integer.

(i) Describe the 3 types of elementary matrices in $F^{n\times n}$.

(ii) If $F = \mathbb{F}_3$ is the prime field with 3 elements, list the number of elementary matrices in $F^{4\times 4}$ of each of the 3 types of part (i).

(iii) How many permutation matrices are there in $F^{4\times 4}$?

(18) 9. Let V be a finite-dimensional vector space over the field $\mathbb{C}$ of complex numbers.

(i) Explain how the concepts "linear operator on V" and "V is a module over the polynomial ring $\mathbb{C}[t]$" are equivalent concepts.

(ii) If $(v_1, \cdots, v_m)$ are generators for V as a $\mathbb{C}[t]$-module, what does it mean for $A \in \mathbb{C}[t]^{m\times n}$ to be a presentation matrix for V with respect to $(v_1, \ldots, v_m)$?

(iii) If $A = \begin{bmatrix} t^2(t-1)^2 & & \\ & t(t-1)(t-2)^2 & \\ & & t(t-2)^3 \end{bmatrix}$ is a presentation matrix for V with respect to (v_1, v_2, v_3), what is the Jordan form of the associated linear operator?

(10) 10. Let $V = \mathbb{C}^3$ and let $T : V \to V$ be a linear operator associated to the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(i) Is the corresponding $\mathbb{C}[t]$-module cyclic? Explain.

(ii) How many 2-dimensional T-invariant subspaces does T have. Describe them.

(10) 11. Let $V = \mathbb{C}^3$ and let $T : V \to V$ be a linear operator associated to the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

(i) Is the corresponding $\mathbb{C}[t]$-module cyclic? Explain.

(ii) How many 2-dimensional T-invariant subspaces does T have? Describe them.

(8) 12. Describe a linear operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ for which there exist infinitely many T-invariant subspaces of dimension 2, but no T-invariant subspaces of dimension 3.

(10) 13. Let $V = \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

(i) How many cyclic subgroups with 9 elements does V have?

(ii) How many noncyclic subgroups with 9 elements does V have?

(16) 14. Assume that $A \in \mathbb{C}^{5\times 5}$ is such that $\operatorname{trace}(A^i) = 0,\ i = 1, 2, 3, 4, 5$. Prove or disprove that A must be nilpotent.