Qualifying Examination
August, 1996
Math 554

In answering any part of a question you may assume the preceding parts.

Notation: K is a field; $M_n(K)$ is the set of $n \times n$ matrices with elements from K; V is an n-dimensional vector space over K; α is a linear operator on V.

1. Prove that if $A, B \in M_n(K)$ and one of A, B is invertible, then $\det(aA+B)=0$ for at most n distinct values of $a \in K$. [8 points]

2. Let A^T denote the transpose of $A \in M_n(K)$. Prove that there exists an invertible $P \in M_n(K)$, such that $PAP^{-1} = A^T$. [8 points]

3. Let π_1 and π_2 be linear operators on a vector space V, such that

$$\pi_1\pi_2 = \pi_2\pi_1 \qquad \pi_1^2 = \pi_1 \qquad \pi_2^2 = \pi_2\,.$$

Prove that V is the direct sum of the following four subspaces: [8 points]

$$\operatorname{Im}\pi_1 \cap \operatorname{Im}\pi_2 \qquad \operatorname{Im}\pi_1 \cap \operatorname{Ker}\pi_2 \qquad \operatorname{Ker}\pi_1 \cap \operatorname{Im}\pi_2 \qquad \operatorname{Ker}\pi_1 \cap \operatorname{Ker}\pi_2\,.$$

4. Prove that if α has the same matrix in all bases of V, then there exists an $a \in K$ such that $\alpha = a\,\mathrm{id}_V$. [8 points]

5. Prove that if $\operatorname{rank}(\alpha) = 1$, then the minimal polynomial of α has the form $x(x-a)$ for some $a \in K$. [8 points]

6. Let $K = \mathbb{R}$ and let V be a space with inner product $(\ |\ )$. If $\alpha \neq 0$ and $(\alpha(v)|w) = -(v|\alpha(w))$ for all $v, w \in V$, then prove the following:

(1) There exists an invariant subspace W of V, with orthonormal basis e_1, e_2, such that $\alpha(e_1) = -e_2$ and $\alpha(e_2) = e_1$. [8 points]

(2) The orthogonal complement $W^\perp$ of W is α-invariant. [6 points]

(3) There exists an orthonormal basis of V in which the matrix of α has the form

$$\begin{bmatrix} A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_k & 0 \\ 0 & 0 & \dots & 0 & O_{n-2k} \end{bmatrix} \qquad \text{where} \quad A_i = \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix} \quad \text{with} \quad a_i \in K$$

and O_{n-2k} is the zero matrix of order $n-2k$. [6 points]

7. Let $\beta\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ be a homomorphism of abelian groups, given by left multiplication with the matrix $\begin{bmatrix} -1 & 3 & 2 \\ 0 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix}$.

(1) Explain why $\operatorname{Ker}\beta$ is a free abelian group, and find a basis. [8 points]

(2) Decompose $\mathbb{Z}^3/\operatorname{Im}\beta$ as a direct sum of cyclic groups. [8 points]

8. Let p be a prime number, and $A = \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^3)$. Compute:

(1) The number of elements of A of order p^2. [8 points]

(2) The number of cyclic subgroups of A of order p^2. [8 points]

(3) The number of non-cyclic subgroups of A of order p^2. [8 points]