QUALIFYING EXAMINATION

August 1998

MATH 554 - Profs. Heinzer/Matsuki

Name: ______________________________

(12) 1. Give an example of an infinite dimensional vector space V over the field of real numbers $\mathbb{R}$ and linear operators S and T on V such that

(i) S is onto, but not one-to-one.

(ii) T is one-to-one, but not onto.

(10) 2. Let $\mathbb{Q}$ denote the field of rational numbers. Give an example of a linear operator $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ having the property that the only T-invariant subspaces are the whole space and the zero subspace. Explain why your example has this property.

(18) 3. Let A and B be $n \times n$ matrices over the field $\mathbb{Q}$ of rational numbers.

(i) Define "A and B are similar over $\mathbb{Q}$".

(ii) True or False: "If A and B are similar over the field $\mathbb{C}$ of complex numbers, then A and B are also similar over $\mathbb{Q}$." Justify your answer.

(iii) Let M and N be $n \times n$ matrices over the polynomial ring $\mathbb{Q}[t]$. Define "M and N are equivalent over $\mathbb{Q}[t]$".

(iv) True or False: "Every matrix $M \in \mathbb{Q}[t]^{n\times n}$ is equivalent over $\mathbb{Q}[t]$ to a diagonal matrix." Justify your answer.

(18) 3. (continued) Let $I \in \mathbb{Q}[t]^{n\times n}$ be the identity matrix and let $A, B \in \mathbb{Q}^{n\times n}$.

(v) True or False: "If A and B are similar over $\mathbb{Q}$, then $tI - A$ and $tI - B$ are equivalent over $\mathbb{Q}[t]$." Justify your answer.

(vi) True or False: "If $\det(tI - A) = \det(tI - B)$ in $\mathbb{Q}[t]$, then A and B are similar over $\mathbb{Q}$." Justify your answer.

(vii) True or False: "Every invertible matrix $A \in \mathbb{Q}^{n\times n}$ is similar to a diagonal matrix over $\mathbb{C}$." Justify your answer.

(18) 4. Let $\mathbb{Z}$ denote the ring of integers and let n be a positive integer. Prove that if M is a submodule of the free $\mathbb{Z}$-module $\mathbb{Z}^n$, then M is a free $\mathbb{Z}$-module.

(18) 5. Let V be a finite-dimensional vector space over an algebraically closed field F and let $T : V \to V$ be a linear operator. Prove that $T = D + N$, where D is a diagonalizable linear operator and N is a a nilpotent linear operator and where D and N are polynomials in T.

(14) 6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator left multiplication by $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

True or False: "If W is a T-invariant subspace of $\mathbb{R}^3$, then there exists a T-invariant subspace W' of $\mathbb{R}^3$ such that $W \oplus W' = \mathbb{R}^3$." Justify your answer.

(16) 7. Let $F = \mathbb{F}_7$ be a finite field with 7 elements.

(i) What is the order of the multiplicative group $GL_2(\mathbb{F}_7)$ of 2×2 invertible matrices with entries from $\mathbb{F}_7$?

(ii) What is the order in $GL_2(\mathbb{F}_7)$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

(iii) What is the order in $GL_2(\mathbb{F}_7)$ of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$?

(iv) What is the order of the group $SL_2(\mathbb{F}_7)$ of matrices in $GL_2(\mathbb{F}_7)$ having determinant 1?

(10) 8. Let $R = F[t]$ denote a polynomial ring in one variable over a field, let n be a positive integer, and let V be a free R-module of rank n.
True or False: "If W is a proper submodule of V, then W is a free R-module of rank $m < n$." Justify your answer.

(10) 9. Let F be a field and let $V = F^{4\times 4}$ be the vector space of 4×4 matrices over F. For $A \in F^{4\times 4}$, define $T_A : V \to V$ by $T_A(B) = AB$ for each $B \in V$.
True or False: "The minimal polynomial of T_A is never equal to the characteristic polynomial of T_A." Justify your answer.

(18) 10. Let F be a field, let m and n be positive integers and let $A \in F^{m\times n}$ be an $m \times n$ matrix.

(i) Define "row space of A".

(ii) Define "column space of A".

(iii) Prove that the dimension of the row space of A is equal to the dimension of the column space of A.

(12) 11. Let $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^3$ be defined by left multiplication by the matrix $\begin{bmatrix} 2 & 6 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}$ and let $V = \mathbb{Z}^3/\varphi(\mathbb{Z}^2)$. Decompose V as a direct sum of cyclic abelian groups.

(12) 12. Assume that $A \in \mathbb{R}^{3\times 3}$ has eigenvalues $0, 2, 4$ and that v_0, v_2, v_4 are associated eigenvectors.

(i) Determine a basis for the column space of A?

(ii) Determine all solutions of the system of equations $AX = v_2 + v_4$.

(14) 13. Let t be an indeterminate over the field $\mathbb{R}$ and let $\varphi : \mathbb{R}[t]^3 \to \mathbb{R}[t]^3$ be defined by left multiplication by the matrix $\begin{bmatrix} t(t-1) & 0 & 0 \\ 2 & t(t-1)^2 & 0 \\ 0 & 0 & t^3(t-1) \end{bmatrix}$. Decompose $\mathbb{R}[t]^3/\varphi(\mathbb{R}[t]^3)$ as a direct sum of cyclic $\mathbb{R}[t]$-modules.