Math 554 - Heinzer **Qualifying Exam** **August 1999**

Name: ____

(20) 1. Let $A \in \mathbb{C}^{4\times 4}$ be a diagonal matrix with main diagonal entries $1, 2, 3, 4$. Define $T_A : \mathbb{C}^{4\times 4} \to \mathbb{C}^{4\times 4}$ by $T_A(B) = AB - BA$.

(i) What is $\dim(\ker(T_A))$?

(ii) What is $\dim(\text{im}(T_A))$?

(iii) What are the eigenvalues of T_A?

(iv) What is the minimal polynomial of T_A?

(v) Is T_A diagonalizable? Explain.

(12) 2. (i) Let $A \in \mathbb{Z}^{3\times 4}$ and define $\phi_A : \mathbb{Z}^4 \to \mathbb{Z}^3$ by $\phi_A(X) = AX$.

True or False? If ϕ_A is surjective, then the determinant of some 3×3 minor of A is a unit of $\mathbb{Z}$. Explain.

(ii) Let $B \in \mathbb{Z}^{4\times 3}$ and define $\phi_B : \mathbb{Z}^3 \to \mathbb{Z}^4$ by $\phi_B(X) = BX$.

True or False? If the determinant of some 3×3 minor of B is nonzero, then ϕ_B is injective. Explain.

(10) 3. True or False? If $A \in \mathbb{R}^{n\times n}$ is normal and if the eigenvalues of A are all real, then A is symmetric. Justify your answer.

(12) 4. Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

(10) 5. Let V be a vector space over an infinite field F. Suppose $\alpha_1, \ldots, \alpha_m$ are finitely many nonzero vectors in V. Prove there exists a linear functional f on V such that $f(\alpha_i) \neq 0$ for each i.

(20) 6. Let V be an abelian group generated by a, b, c, where $2a = 4b, 2b = 4c, 2c = 4a$, and where these 3 relations generate all the relations on a, b, c.

(i) For some positive integer n, find elements $x_1, \ldots, x_n \in V$ that generate V and have the property that $c_i \in \mathbb{Z}$ with $c_1x_1 + c_2x_2 + \cdots c_nx_n = 0$ implies each $c_ix_i = 0$.

(ii) Write V as a direct sum of cyclic groups. What is the order of V?

(18) 7. Let F be a field, let m and n be positive integers, and let $F^{m\times n}$ denote the

set of $m \times n$ matrices with entries in F.

(i) What does it mean for $R \in F^{m\times n}$ to be a row-reduced echelon matrix ?

(ii) Suppose W is a subspace of F^n with $\dim W \leq m$. Prove there is precisely one row-reduced echelon matrix $R \in F^{m\times n}$ such that W is the row space of R.

(12) 8. Suppose F is a field of characteristic zero and V is a finite-dimensional vector space over F. If $E_1, \dots, E_k$ are projection operators of V such that $E_1 + \cdots + E_k = I$, the identity operator on V, prove that $E_iE_j = 0$ for $i \neq j$.

(10) 9. Prove or disprove: if $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear operator such that every subspace of $\mathbb{R}^4$ is invariant under T, then T is a scalar multiple of the identity operator.

(18) 10. Suppose $\mathcal{F}$ is a subspace of $\mathbb{C}^{4\times 4}$ such that for each $A, B \in \mathcal{F}$, $AB = BA$. If there exists $A \in \mathcal{F}$ having at least two distinct characteristic values, prove that $\dim \mathcal{F} \leq 4$.

(22) 11. Assume that V is a finite-dimensional vector space over an infinite field F and $T : V \to V$ is a linear operator. Give to V the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(1) Outline a proof that V is a direct sum of cyclic $F[x]$-modules.

(2) In terms of the expression for V as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that V have only finitely many T-invariant submodules? Explain.

(18) 12. Assume that M is a module over an integral domain D. Recall that a submodule N of M is said to be *pure* if for each $y \in N$ and $a \in D$, $ax = y$ is solvable in M if and only if it is solvable in N.

(1) If N is a direct summand of M, prove that N is pure in M

(2) For $x \in M$, let $x + N$ denote the coset representing the image of x in the quotient module M/N. If N is a pure submodule of M and $\operatorname{ann}(x + N)$ is a principal ideal (d) of D, prove that there exists $x' \in D$ such that $x + N = x' + N$ and $\operatorname{ann} x' = \{a \in D \,|\, ax' = 0\}$ is the principal ideal (d).

(18) 13 Assume that M is a finitely generated torsion module over the polynomial ring $F[x]$, where F is a field, and that N is a pure submodule of M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.