Math 554 - Heinzer/Roche **Qualifying Exam** **January 1999**

Name: ______________________________

(7) 1. Let V be an abelian group and assume that $(v_1, \dots, v_m)$ are generators of V. Describe a process for obtaining an $m \times n$ matrix $A \in \mathbb{Z}^{m\times n}$ such that if $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$ is the $\mathbb{Z}$-module homomorphism defined by left multiplication by A, then $V \cong \mathbb{Z}^m/\phi(\mathbb{Z}^n)$. Such a matrix A is called a presentation matrix of V.

(15) 2. Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(1) Write down a presentation matrix for V as a $\mathbb{Z}$-module.

(2) Let W be the cyclic subgroup of V generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z} = V$. Write down a presentation matrix for W.

(3) Write down a presentation matrix for the quotient module V/W.

(20) 3. Let R be a commutative ring and let V and W denote free R-modules of rank 4 and 5, respectively. Assume that $\phi : V \to W$ is an R-module homomorphism, and that $\mathbf{B} = (v_1, \dots, v_4)$ is an ordered basis of V and $\mathbf{B}' = (w_1, \dots, w_5)$ is an ordered basis of W.

(1) What is meant by the coordinate vector of $v \in V$ with respect to the basis $\mathbf{B}$?

(2) Describe how to obtain a matrix $A \in R^{5\times 4}$ so that left multiplication by A on R^4 represents $\phi : V \to W$ with respect to $\mathbf{B}$ and $\mathbf{B}'$.

(3) How does the matrix A change if we change the basis $\mathbf{B}$ by replacing v_1 by $v_1 + v_2$?

(4) How does the matrix A change if we change the basis $\mathbf{B}'$ by replacing w_1 by $w_1 + w_2$?

(18) 4. Let A be an 4×5 matrix with integer coefficients and let $\phi : \mathbb{Z}^5 \to \mathbb{Z}^4$ be defined by left multipliation by A.

(1) Prove or disprove: if ϕ is surjective, then the determinants of the 4×4 minors of A generate the unit ideal of $\mathbb{Z}$.

(2) Prove or disprove: if ϕ is surjective, then there exists a matrix $B \in \mathbb{Z}^{5\times 4}$ such that AB is the 4×4 identity matrix.

(10) 5. Let $V = \mathbb{Z}^2$ and let L be the submodule of V spanned by the columns of $A = \begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix}$. Find a basis $(\vec{\alpha}_1, \vec{\alpha}_2)$ of V and integers c_1, c_2 so that $c_1\vec{\alpha}_1$, $c_2\vec{\alpha}_2$ is a basis for L.

(10) 6. If $A \in \mathbb{R}^{n\times n}$ is symmetric, does there exist $B \in \mathbb{R}^{n\times n}$ such that $B^3 = A$? Justify your answer.

(16) 7. Let F be a field and let $F[t]$ be a polynomial ring in one variable over F. Let r and s and $a_1 \geq a_2 \geq \cdots \geq a_r$ and $b_1 \geq b_2 \geq \cdots \geq b_s$ be positive integers. Suppose

$$V = F[t]/(t^{a_1}) \oplus F[t]/(t^{a_2}) \oplus \cdots \oplus F[t]/(t^{a_r})$$

and

$$W = F[t]/(t^{b_1}) \oplus F[t]/(t^{b_2}) \oplus \cdots \oplus F[t]/(t^{b_s}).$$

If the $F[t]$-modules V and W are isomorphic, prove the structure theorem that asserts that $r = s$, and that $a_i = b_i$ for $i = 1, \ldots, r$.

(10) 8. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and let $f(x) \in \mathbb{C}[x]$ be a monic polynomial. Suppose $a \in \mathbb{C}$ is an eigenvalue of $f(T)$. Prove or disprove that there must exist an eigenvalue b of T such that $f(b) = a$.

(10) 9. Suppose $A \in \mathbb{R}^{5\times 3}$ has rank 3. Let A^T denote the transpose of A. Prove or disprove that $A^TA \in \mathbb{R}^{3\times 3}$ must be nonsingular.

(12) 10. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be a linear operator that preserves orthogonality, i.e., if $u \perp v$, then $T(u) \perp T(v)$. Prove that $T = \lambda S$ for some orthogonal operator S and some $\lambda \in \mathbb{R}$.

(10) 11. Suppose $A \in \mathbb{R}^{n\times n}$. If A is normal and if the eigenvalues of A are all real, does it follow that A is symmetric? Justify your answer with a proof or a counterexample.

(10) 12. If v is a nonzero vector in $\mathbb{R}^3$ and w is a nonzero vector in $\mathbb{R}^5$, must there exist a 3 by 5 matrix A whose column space is spanned by v and whose row space is spanned by w? Justify your answer.

(10) 13. Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

(10) 14. Suppose $S : \mathbb{C}^5 \to \mathbb{C}^5$ and $T : \mathbb{C}^5 \to \mathbb{C}^5$ are commuting linear operators. Prove that there exists a nonzero $v \in \mathbb{C}^5$ which is an eigenvector for both S and T.

(10) 15. If A and B in $\mathbb{R}^{n\times n}$ are normal matrices, does it follow that AB is also normal? Justify your answer.

(10) 16. Is $A \in \mathbb{C}^{n\times n}$ always similar to its transpose A^T? Justify your answer.

(12) 17. Classify up to similarity all matrices $A \in \mathbb{C}^{3\times 3}$ such that $A^3 = I$, i.e., write down all possibilities for the Jordan form of A.