QUALIFYING EXAMINATION
AUGUST 2001
MATH 554 - PROF. ARAPURA

1. (*30 points*) Determine whether the following statements are true or false (you must justify the answer with either a proof or a counterexample). All matrices and vector spaces in this problem are defined over the field $\mathbb{R}$.

a) Let A and B be $n \times n$ matrices. Then AB is invertible if and only if A and B are.

b) Every square matrix is a product of elementary matrices.

c) There exists a 3×2 matrix A and a 2×3 matrix B such that AB is invertible.

d) If V and W are finite dimensional vector spaces such that $dimV \leq dimW$, then V is isomorphic to a subspace of W.

e) If v_1, v_2, v_3 are three distinct nonzero vectors in a finite dimensional vector space V, there exists a linear transformation $f : V \to \mathbb{R}$ satisfying $f(v_i) = i$ for all i.

f) If v_1, v_2, v_3 are three linearly independent vectors in a finite dimensional vector space V, there exists a linear transformation $f : V \to \mathbb{R}$ satisfying $f(v_i) = i$ for all i.

2. (*10 points*) Let V be a finite dimensional vector space. Suppose that $W_1, W_2 \subset V$ are subspaces. Define the linear tranformation $L : W_1 \oplus W_2 \to V$ by $L(w_1, w_2) = w_1 + w_2$. Calculate the kernel and image of L, and use this to prove that

$$dim(W_1 + W_2) = dimW_1 + dimW_2 - dim(W_1 \cap W_2)$$

3. (*20 points*) Let A and B be $n \times n$ matrices over a field F. Suppose that A is invertible.

a) If F is infinite, prove that there exists $\lambda \in F$ such that $\lambda A + B$ is invertible.

b) Give an example to show that the conclusion of part a) can fail when $F = \mathbb{Z}/2\mathbb{Z}$.

4. (*20 points*) Let A be an $n \times n$ matrix with entries in a field F. Assume that A is idempotent i.e. $A^2 = A$. Let $L : F^n \to F^n$ be the corresponding linear transformation defined by $Lv = Av$.

a) Prove that the only possible eigenvalues for A are 0 and 1.

b) Prove that v is an eigenvector of A with eigenvalue 1 if and only if v lies in the image $im(L)$.

c) Prove that $F^n = ker(L) + im(L)$ and that $ker(L) \cap im(L) = 0$.

d) Let $B = (b_{ij})$ be the matrix representing L in a given basis $v_1, \dots v_n$ of F^n, i.e. $Lv_i = \sum_j b_{ji} v_j$. Show that the basis can be chosen so that

$$B = \begin{pmatrix} 0 & & \dots & & 0 \\ & \ddots & & & \\ \vdots & & 0 & & \vdots \\ & & & 1 & \\ & & & & \ddots \\ 0 & & \dots & & 1 \end{pmatrix}$$

5. (*10 points*) Let $S \subset \mathbb{Z}^3$ be the sub-abelian group generated by $(2,2,2)^T$ and $(3,1,1)^T$. Express $\mathbb{Z}^3/S$ as a direct sum of cyclic groups.

6. (*10 points*) Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

and let $M = \mathbb{C}^3$ with the $\mathbb{C}[x]$-module structure determined by the rule $p(x)v = p(A)v = (a_n A^n + \dots + a_0 I)v$ for $p(x) = a_n x^n + \dots + a_0 \in \mathbb{C}[x]$, $v \in M$. Find a polynomial $f(x)$ such that M is isomorphic to $\mathbb{C}[x]/(f)$. (Hint: Consider the $\mathbb{C}[x]$-module homomorphism $\phi : \mathbb{C}[x] \to M$ which sends 1 to $(1,0,0)^T$.)

7. (*20 points*) Suppose that A is a 2×2 matrix over an algebraically closed field F.

a) Prove that A is either diagonalizable or similar to a matrix of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, and show that these possibilities are mutually exclusive.

b) Prove that A^2 is always diagonalizable if F has characteristic 2.