QUALIFYING EXAM

Math 554

August 2003

1. (12 points) Without proof give an example of:

(a) a domain R and a finitely generated torsionfree R-module that is not free;

(b) a nonzero commutative ring R that is not a field so that every R-module is torsionfree;

(c) a normal matrix with entries in $\mathbb{R}$ that is not similar to a diagonal matrix with entries in $\mathbb{R}$.

2. (13 points) For R a commutative ring and M an R-module let $M^\star = \mathrm{Hom}_R(M, R)$ denote the R-dual of M.

(a) Let M and N be R-modules. Show that $(M \oplus N)^\star \cong M^\star \oplus N^\star$.

(b) Let R be a principal ideal domain and M a finitely generated R-module. Show that $M^\star$ is a free R-module with $\operatorname{rank} M^\star = \operatorname{rank} M$.

3. (12 points) Let $\mathbb{Q}[X]$ be the polynomial ring in one variable over $\mathbb{Q}$, n a positive integer, and $R = \mathbb{Q}[X]/(X^n)$. Classify all finitely generated R-modules up to R-isomorphisms.

4. (15 points) Let R be a commutative ring, F a free R-module of finite rank, and $\varphi \in \mathrm{End}_R(F)$ an R-endomorphism of F. Show that the following are equivalent:

(a) φ is bijective;

(b) φ is surjective;

(c) $\det(\varphi)$ is a unit of R.

5. (20 points) Let K be a field, V a K-vector space of dimension n, $\mathrm{End}_K(V)$ the K-vector space of K-endomorphisms of V, and $\varphi \in \mathrm{End}_K(V)$ a fixed K-endomorphism. Show that:

(a) $U = \{\psi \in \mathrm{End}_K(V) |\, \varphi\psi = \psi\varphi\}$ is a subspace of $\mathrm{End}_K(V)$;

(b) $\dim_K U \geq n$;

(hint: first consider the case where V has a basis of the form $\{\varphi^i(v) | \, 0 \leq i \leq n-1\}$ for some $v \in V$)

(c) $W = \{\varphi\psi - \psi\varphi | \, \psi \in \mathrm{End}_K(V)\}$ is a subspace of $\mathrm{End}_K(V)$;

(d) $\dim_K W \leq n^2 - n$.

6. (13 points) Determine the rational canonical form of the following matrix with entries in $\mathbb{Q}$:

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 2 & -2 & 1 & -3 & -2 \\ 0 & -2 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

7. (15 points) Let K be a field of characteristic zero, $V = \mathrm{Mat}_n(K)$ the K-vector space of n by n matrices with entries in K, and $W = \{A \in V | \, \mathrm{Tr}(A) = 0\}$ the subspace of V consisting of the matrices whose trace is zero. Let $f : V \times V \to K$ be defined by $f(A, B) = n\,\mathrm{Tr}(AB) - \mathrm{Tr}(A)\,\mathrm{Tr}(B)$.

(a) Show that f is a symmetric bilinear form;

(b) show that f restricted to W is non-degenerate;

(c) determine $V^{\perp}$ and the rank of f.