QUALIFYING EXAMINATION

AUGUST 2005

MATH 554 - Dr. C. Wilkerson

There are eight problems, each worth 25 points for a total of 200 points.Unless otherwise stated, show all necessary work. All rings are assumed to be commutative rings with a multiplicative identity element.

I. (a) Let A be a finite abelian group of order $9 * 256$. Let $\phi_n : A \to A$ be the group homomorphism that sends $x \to nx$, for any integer n. The following information is known about $\ker(\phi_n)$

n	$\#\ker(\phi_n)$	$\#\ker(\phi_n^2)$	$\#\ker(\phi_n^3)$
2	8	64	256
3	3	9	9

Deduce the structure of A as a direct sum of cyclic groups of prime power order. Give the invariant factors for A.

(b) Let V be an 8 dimensional vector space over a field K and let $\psi \in \mathrm{End}_K(V)$. Suppose that the kernel of $(\psi - 5)^j$ has dimension k over K and that the following is known about k: for $j = 1$, $k = 4$; for $j = 2$, $k = 7$, and for $j = 3$, $k = 8$. Write down the rational canonical form and Jordan canonical form for ψ.

II. (a) Define the concepts of Euclidean domain, PID, and UFD.

(b) Suppose that R is a Euclidean domain. Prove that R is a PID.

(c) Give an example of a UFD that is not a PID.

III. (a) Give an example of a ring R and a finitely generated module over R that is torsion free, but not free.

(b) Prove that a finitely generated module over a PID that is torsion free is free.

(c) If M is an R-module, show that $\mathrm{Hom}_R(M, R)$ is torsion free.

(d) If R is a ring and M a module over R, define $\mathrm{Qtor}(M) = \{m \in M | \text{there is } r \neq 0 \in R \text{ such that } rm = 0\}$. Give an example to show that if R is not a domain, then $\mathrm{Qtor}(M)$ need not be a submodule of M.

IV. Without proof, give examples of the following:

a) A submodule of a module which is not a direct summand.

b) A symmetric bilinear form on a finite dimensional vector space that is not diagonalizable.

c) A normal matrix over the reals that is not diagonalizable.

d) A matrix over the complex numbers that is not diagonalizable.

V. Let R be a PID and $0 \to N \to M \to Q \to 0$ an exact sequence of R-modules, where M is finitely generated. Show that the following two statements are equivalent:

a) M is torsion free and the exact sequence is split exact.

b) N and Q are torsion free.

VI. Find the eigenvalues, characteristic polynomial, minimal polynomial, rational canonical form and Jordan canonical form in $\mathrm{Mat}_4(\mathbb{C})$ of

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & 1 & -1 \\ -2 & 1 & 0 & -1 \end{pmatrix}$$

VII. Let $(V, < \, , \, >)$ be a finite dimensional inner product space over $K = \mathbb{R}$ or $\mathbb{C}$, and $\phi \in \mathrm{End}_K(V)$.

(a) define the adjoint ϕ^T of ϕ.

(b) define the normal and self-adjoint properties for such ϕ.

(c) show that $\ker(\phi^T) = \mathrm{im}(\phi)^\perp$, and if ϕ is normal, also that $\ker(\phi) = \mathrm{im}(\phi)^\perp$.

VIII. Let R be a ring and let A and B be in $\mathrm{Mat}_n(R)$ so that

$$AB = aI_n$$

for some non-zerodivisor $a \in R$. Show that $AB = BA$.