QUALIFYING EXAMINATION

Math 554

January 2005 - Prof. Ulrich

1. (12 points)

Without proof give an answer to these questions:

(a) For R a commutative ring and M an R-module, is the R-module $\operatorname{Hom}_R(M, R)$ torsion free?

(b) How many isomorphism classes are there of $\mathbb{Z}_{20}$-modules having exactly 625 elements?

(c) How many isometry classes are there of alternating bilinear forms on a 3-dimensional vector space?

2. (15 points)

Let R be a commutative ring and P an R-module. Prove that the following are equivalent:

(a) For every R-linear map $f : P \to N$ and every R-epimorphism $\pi : M \to N$ there exists an R-linear map $g : P \to M$ with $\pi g = f$.

(b) There exists an R-module Q such that the direct sum $P \oplus Q$ is a free R-module.

3. (15 points)

Let R be an integral domain and F a free R-module with ordered basis $\{x_1, \dots, x_n\}$. Let $M = Ru_1 + \dots + Ru_n \subset N = Rv_1 + \dots + Rv_n$ be submodules of F with $u_i = \sum_j a_{ij}x_j$, $v_i = \sum_j b_{ij}x_j$ $(a_{ij}, b_{ij} \in R)$, and consider the n by n matrices $A = (a_{ij})$, $B = (b_{ij})$.

(a) Prove that M and N are free R-modules if the determinant $\det A \neq 0$.

(b) Assume $\det A \neq 0$. Prove that $M = N$ if and only if $\det A$ and $\det B$ are associates.

4. (13 points)

Determine whether the matrices

$$\begin{bmatrix} 2 & -4 & 14 & 10 \\ -2 & 7 & 4 & 5 \\ 1 & -2 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -4 & 5 & 0 & 7 \\ -2 & 4 & 12 & 2 \\ -2 & 4 & 6 & 8 \end{bmatrix}$$

are equivalent over $\mathbb{Z}$. Show your work.

5. (16 points)

Consider the ring $R = \mathbb{Q}[X]/((X^4+2)(X+1)^2)$ as a $\mathbb{Q}[X]$-module, and let φ be the $\mathbb{Q}[X]$-endomorphism of R defined by $\varphi(a) = X^2 \cdot a$. Determine the rational canonical form of φ considered as a $\mathbb{Q}$-linear map.

6. (13 points)

Let R be a principal ideal domain, let M be a finitely generated R-module with a symmetric bilinear form f, and write

$$M^{\perp} = \{ x \in M \mid f(x,y) = 0 \text{ for every } y \in M \}$$

for the orthogonal complement of M in M. Show that $M = F \oplus M^{\perp}$ for some free submodule F of M.

7. (16 points)

Let A and B be symmetric n by n matrices with entries in $\mathbb{R}$. Show that $AB = BA$ if and only if there exists an orthogonal matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that both PAP^t and PBP^t are diagonal matrices.