QUALIFYING EXAMINATION

MATH 554, August 2006

Prof. J-K Yu and Prof. J. Wang

There are 6 problems with a total of 12 parts. Each part is worth 10 points. You can do 4(b) by assuming 4(a), and so on.

1. (a) Let $p_A(t)$ denote the characteristic polynomial of an $n \times n$ matrix A, i.e. $p_A(t) = \det(tI_n - A)$. Let A be a complex $n \times n$ matrix and $f(T)$ be a polynomial in T with complex coefficients. Show that $p_{f(A)}(t)$ is determined by $f(T)$ and $p_A(t)$.

(b) Now suppose that A is 3×3 satisfying $A^3 + A + I_3 = 0$ with coefficients in $\mathbf{Q}$. Find the characteristic polynomial of $A^2 + I_3$. You may use the fact that the polynomial $t^3 + t + 1$ is irreducible over $\mathbf{Q}$.

2. (a) Let A be an $n \times n$ invertible matrix over $\mathbf{C}$ and $m \geq 1$ an integer. Show that if A^m is diagonalizable, then so is A. (*Hint.* Consider a normal form).

(b) Show that (a) fails if $\mathbf{C}$ is replaced by a field of characteristic $p > 0$.

3. Let A be a real anti-symmetric square matrix, i.e. $A^t = -A$. Show that the eigenvalues of A are purely imaginary (i.e. of the form it with $t \in \mathbf{R}$). (*Hint.* Recall the algebraic proof of the fact that symmetric real matrices have real eigenvalues).

4. (a) Let A be an $n \times n$ complex matrix and V the vector space of $n \times n$ complex symmetric matrices, so that $\dim V = n(n+1)/2$. Let $L = L_A : V \to V$ be the linear map defined by $L(X) = AXA^t$. Suppose that A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that the eigenvalues of L are $\{\lambda_i\lambda_j : 1 \leq i \leq j \leq n\}$.

(b) Show that the above remains true for any A, diagonalizable or not. (*Hint. First Approach:* Show that if $A = S + N$ is the Jordan decomposition, then L_S is the semisimple part of L_A. *Second approach:* Show that it is enough to consider an upper triangular A, and choose a suitable basis for V. There are other approaches).

5. (a) Let P_n be the $(n+1)$-dimensional vector space of *homogeneous* real polynomials in x, y of degree n. Fix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $L : P_n \to P_n$ by $L(f(x,y)) = f(ax+by, cx+dy)$. Show that L is a linear map.

(b) Now let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and consider the linear map $N : P_n \to P_n$ defined by $N(f) = L(f) - f$. Find $N(y^n)$, $N^2(xy^{n-1})$, …, $N^{n+1}(x^n)$.

(c) Find the Jordan form of N. (*Hint.* (b) tells you what N^{n+1} is. Now it remains to determine what N^n is (or rather is not)).

6. (a) Let A be an $n \times n$ real matrix. Let $v_1, \ldots, v_n \in \mathbf{R}^n$ be the column vectors of A. Show that

$$|\det(A)| \leq ||v_1|| \cdot ||v_2|| \cdots ||v_n||,$$

where $||v||$ is the standard norm of $v \in \mathbf{R}^n$. (*Hint.* Write $A = QT$ with Q orthogonal and T upper triangular).

(b) Let B be a positive symmetric $n \times n$ real matrix with diagonal entries $d_1, \ldots, d_n$ (i.e. if $B = (b_{ij})$ then $d_i = b_{ii}$). Show that

$$\det(B) \leq d_1 \cdot d_2 \cdots d_n.$$

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There are 6 problems with a total of 12 parts. Each part is worth 10 points. You can do 4(b) by assuming 4(a), and so on.

1. (a) Let $p_A(t)$ denote the characteristic polynomial of an $n \times n$ matrix A, i.e. $p_A(t) = \det(tI_n - A)$. Let A be a complex $n \times n$ matrix and $f(T)$ be a polynomial in T with complex coefficients. Show that $p_{f(A)}(t)$ is determined by $f(T)$ and $p_A(t)$.

(b) Now suppose that A is 3×3 satisfying $A^3 + A + I_3 = 0$ with coefficients in $\mathbf{Q}$. Find the characteristic polynomial of $A^2 + I_3$. You may use the fact that the polynomial $t^3 + t + 1$ is irreducible over $\mathbf{Q}$.

Solution. (a) It is easy to see that if A is similar to A' then $f(A)$ is similar to $f(A')$. Therefore, we may and do replace A by a matrix similar to A and hence assume that A is upper triangular, with diagonal entries $\lambda_1, \ldots, \lambda_n$. It follows that $f(A)$ is also upper triangular with diagonal entries $f(\lambda_1), \ldots, f(\lambda_n)$. Therefore,

$$p_{f(A)}(t) = \prod (t - f(\lambda_i)) \qquad \text{if} \qquad p_A(t) = \prod (t - \lambda_i).$$

(b) Since $t^3 + t + 1$ is irreducible over $\mathbf{Q}$, it is the characteristic polynomial of A, and A is determined by this condition up to similarity. We may and do assume that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \text{ (rational normal form).}$$

A simple calculation then gives

$$A^2 + I_3 = A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad p_{A^2+I}(t) = t^3 - t^2 - 1.$$

2. (a) Let A be an $n \times n$ invertible matrix over $\mathbf{C}$ and $m \geq 1$ an integer. Show that if A^m is diagonalizable, then so is A. (*Hint.* Consider a normal form).

(b) Show that (a) fails if $\mathbf{C}$ is replaced by a field of characteristic $p > 0$.

Solution. (a) The question is unchanged if we replace A by a matrix similar to A. Therefore, we may and do assume that A is in its Jordan form. We may further assume that A consists of only one Jordan block.

Write $A = \lambda I + N$ with $\lambda \neq 0$ being the eigenvalue of A, and N nilpotent. Then $A^m = \lambda^m I + m\lambda^{m-1}N + \binom{m}{2}N^2 + \cdots$. Since $I, N, \ldots, N^{n-1}$ are linearly independent in $M_{n\times n}(\mathbf{C})$, it is clear that A^m is not diagonalizable unless $n = 1$, i.e. unless A itself is diagonalizable.

(b) Take $A = I + N$ with N a non-zero nilpotent matrix. Then $A^p = I$ is diagonalizable, but A is not.

3. Let A be a real anti-symmetric square matrix, i.e. $A^t = -A$. Show that the eigenvalues of A are purely imaginary (i.e. of the form it with $t \in \mathbf{R}$). (*Hint.* Recall the algebraic proof of the fact that symmetric real matrices have real eigenvalues).

Solution. Suppose that λ is an eigenvalue of A with eigenvector $v \neq 0$. Then

$$\lambda(v,v) = (Av,v) = (v,A^t v) = (v,-Av) = -\bar{\lambda}(v,v),$$

where $(-,-)$ is the standard hermitian form. This forces $\lambda = -\bar{\lambda}$ to be purely imaginary.

4. (a) Let A be an $n \times n$ complex matrix and V the vector space of $n \times n$ complex symmetric matrices, so that $\dim V = n(n+1)/2$. Let $L : V \to V$ be the linear map defined by $L(X) = AXA^t$. Suppose that A is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that the eigenvalues of L are $\{\lambda_i\lambda_j : 1 \leq i \leq j \leq n\}$.

(b) Show that the above remains true for any A, diagonalizable or not. (*Hint. First Approach:* Show that if $A = S+N$ is the Jordan decomposition, then L_S is the semisimple part of L_A. *Second approach:* Show that it is enough to consider an upper triangular A, and choose a suitable basis for V. There are other approaches).

Solution. (a) Let $v_1, \dots, v_n$ be an eigenbasis of A with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. It is easy to see (1) $\{v_i v_j^t\}_{i,j}$ forms a basis of $M_{n\times n}(\mathbf{C})$; (2) $\{v_i v_j^t + v_j v_i^t \in V$ is an eigenvector of L with eigenvalue $\lambda_i\lambda_j$, for $1 \leq i \leq j \leq n$; (3) the eigenvectors in (2) are linearly independent by (1).

It follows that the eigenvectors in (2) is an eigenbasis of L, and hence L is diagonalizable with the stated eigenvalues.

(b) There are a few standard tricks to do this. It is fairly easy to follow one of the two approaches given in the hint. Another approach is the following. Let $\prod_{i=1}^n (t-\lambda_i) = \sum_{i=0}^n (-1)^j c_j t^{n-j}$ so that the c_j's are the elementary symmetric polynomials of the λ_i's. Write $\prod_{1\leq i\leq j\leq n}(t-\lambda_i\lambda_j) = \sum d_k t^{n-k}$. By the theory of symmetric polynomials,

$$d_k = P_k(c_1, \dots, c_n)$$

for some universal polynomial P_k's with coefficients in $\mathbf{C}$ (actually in $\mathbf{Z}$). Now we want to prove that the characteristic polynomial of $L = L_A$ can be expressed in terms of that of A using these P_k's. This amounts to a bunch of polynomial identities in n^2 variables (which are the entries of A). By (a), we know that these identities hold on a dense subset of $\mathbf{C}^{n^2}$. Therefore, they hold everywhere.

5. (a) Let P_n be the $(n+1)$-dimensional vector space of *homogeneous* real polynomials in x, y of degree n. Fix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $L : P_n \to P_n$ by $L(f(x,y)) = f(ax+by, cx+dy)$. Show that L is a linear map.

(b) Now let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and consider the linear map $N : P_n \to P_n$ defined by $N(f) = L(f) - f$. Find $N(y^n)$, $N^2(xy^{n-1})$, ..., $N^{n+1}(x^n)$.

(c) Find the Jordan form of N. (*Hint.* (b) tells you what N^{n+1} is. Now it remains to determine what N^n is (or rather is not)).

Solution. (a) is very straightforward. For (b), we compute directly and find $N(y^n) = N^2(xy^{n-1}) = \cdots = N^{n+1}(x^n) = 0$.

(c) By (b), we have $N^{n+1} = 0$. Inductively, we can also establish $N^i(x^i y^{n-i}) \neq 0$ for $i = 0, \dots, n$. Therefore, $N^n \neq 0$. It follows that N is nilpotent with a single $(n+1)\times(n+1)$ Jordan block.

6. (a) Let A be an $n \times n$ real matrix. Let $v_1, \ldots, v_n \in \mathbf{R}^n$ be the column vectors of A. Show that

$$|\det(A)| \leq ||v_1|| \cdot ||v_2|| \cdots ||v_n||,$$

where $||v||$ is the standard norm of $v \in \mathbf{R}^n$. (*Hint.* Write $A = QT$ with Q orthogonal and T upper triangular).

(b) Let B be a positive symmetric $n \times n$ real matrix with diagonal entries $d_1, \ldots, d_n$ (i.e. if $B = (b_{ij})$ then $d_i = b_{ii}$). Show that

$$\det(B) \leq d_1 \cdot d_2 \cdots d_n.$$

Solution. (a) Notice that this has a very intuitive interpretation via volumes, which easily leads to the following formal proof. Use induction on n. The case of $n = 1$ is trivial. Now notice that we may replace A by UA without changing the problem, where U is an orthogonal matrix. Thus we may assume that v_1 is of the form $c \cdot e_1$, $e_1 = (1, 0, \ldots, 0)^t$.

Let A' be the $(n-1) \times (n-1)$ submatrix in the lower right corner of A, and let $v'_2, \ldots, v'_n$ be the column vectors of A'. Notice that $||v'_j|| \leq ||v_j||$ for $j = 2, \ldots, n$. Now we have

$$|\det(A)| = |c| \det(A') \leq |c| \cdot ||v'_2|| \cdots ||v'_n|| \leq ||v_1|| \cdot ||v_2|| \cdots ||v_n||$$

by induction hypothesis. One can also use the hint to reduce directly to the case of an upper triangular A, for which the statement is obvious (really the same proof).

(b) Since B is positive definite, we can write $B = A.A^t$ for a suitable $n \times n$ matrix A (we may take A to be symmetric positive definite; but it doesn't matter here). Now

$$\det(B) = \det(A)^2 \leq ||v_1||^2 \cdots ||v_n||^2 = d_1 \cdots d_n.$$

Here the v_j's are the column vectors of A and we are using (a), noticing that $d_j = ||v_j||^2$.