Math 554 Qualifying Exam Heinzer January 2007

(12) 1. Let F be a field, let n be a positive integer, and let $W = F^{n\times n}$ denote the vector space of $n \times n$ matrices with entries in F.

(i) Let W_0 denote the subspace of W spanned by the matrices C of the form $C = AB - BA$. What is $\dim W_0$?

(ii) For $A \in F^{n\times n}$, define the adjoint matrix $\operatorname{adj} A \in F^{n\times n}$.

(iii) If $A \in \mathbb{R}^{3\times 3}$ and $\det A = 2$, what is $\det \operatorname{adj} A$?

(10) 2. Let $\mathbb{Q}$ denote the field of rational numbers. Give an example of a linear operator $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ having the property that the only T-invariant subspaces are the whole space and the zero subspace. Explain why your example has this property.

(20) 3. Let A and B in $\mathbb{Q}^{n\times n}$ be $n \times n$ matrices and let $I \in \mathbb{Q}^{n\times n}$ denote the identity matrix.

(i) State true or false and justify: If A and B are similar over an extension field F of $\mathbb{Q}$, then A and B are similar over $\mathbb{Q}$.

(ii) Let M and N be $n \times n$ matrices over the polynomial ring $\mathbb{Q}[x]$. Define "M and N are equivalent over $\mathbb{Q}[x]$".

(iii) State true or false and justify: Every matrix $M \in \mathbb{Q}[x]^{n\times n}$ is equivalent to a diagonal matrix.

(iv) State true or false and justify: If $\det(xI - A) = \det(xI - B)$, then $xI - A$ and $xI - B$ are equivalent.

(v) State true or false and justify: If A and B are similar over $\mathbb{Q}$, then $xI - A$ and $xI - B$ are equivalent in $\mathbb{Q}[x]$.

(14) 4. Let F be a field, let m and n be positive integers and let $A \in F^{m \times n}$ be an $m \times n$ matrix.

(i) Define "row space of A".

(ii) Define "column space of A".

(iii) Prove that the dimension of the row space of A is equal to the dimension of the column space of A.

(16) 5. Let D be a principal ideal domain and let V and W denote free D-modules of rank 5 and 4, respectively. Assume that $\phi : V \to W$ is a D-module homomorphism, and that $\mathbf{B} = \{v_1, \ldots, v_5\}$ is an ordered basis of V and $\mathbf{B}' = \{w_1, \ldots, w_4\}$ is an ordered basis of W.

(i) Define what is meant by the coordinate vector of $v \in V$ with respect to the basis $\mathbf{B}$?

(ii) Describe how to obtain a matrix $A \in D^{4 \times 5}$ so that left multiplication by A on D^5 represents $\phi : V \to W$ with respect to $\mathbf{B}$ and $\mathbf{B}'$.

(iii) How does the matrix A change if we change the basis $\mathbf{B}$ by replacing v_2 by $v_2 + av_1$ for some $a \in D$?

(iv) How does the matrix A change if we change the basis $\mathbf{B}'$ by replacing w_2 by $w_2 + aw_1$ for some $a \in D$?

(12) 6. Let $\mathcal{F}$ be a subspace of $\mathbb{C}^{4\times 4}$ of commuting matrices.

(i) Demonstrate with an example that it is possible for there to exist in $\mathcal{F}$ five elements that are linearly independent over $\mathbb{C}$.

(ii) If there exists $A \in \mathcal{F}$ having at least two distinct characteristic values, prove that $\dim \mathcal{F} \leq 4$.

(20) 7. Let V be a finite-dimensional vector space over the field F and let $T : V \to V$ be a linear operator. Give V the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(i) If $\{v_1, \cdots, v_n\}$ are generators for V as an $F[x]$-module, what does it mean for $A \in F[x]^{m\times n}$ to be a relation matrix for V with respect to $\{v_1, \ldots, v_n\}$?

(ii) If $F = \mathbb{C}$ and $A = \begin{bmatrix} x^2(x-1) & 0 & 0 \\ 0 & x(x-1)(x-2) & 0 \\ 0 & 0 & x^2(x-2) \end{bmatrix}$ is a relation matrix for V with respect to $\{v_1, v_2, v_3\}$, list the invariant factors of V.

(iii) With assumptions as in part (ii), list the elementary divisors of V and describe the direct sum decomposition of V given by the primary decomposition theorem.

(iv) With assumptions as in part (ii), write the Jordan form of the operator T.

(8) 8. Let V be a five-dimensional vector space over the field F and let $T : V \to V$ be a linear operator such that $\operatorname{rank} T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of T. Explain.

(8) 9. Let V be an abelian group with generators $\{v_1, v_2, v_3\}$ that has the matrix $\begin{bmatrix} 2 & 0 & 6 \\ 4 & 8 & 0 \end{bmatrix}$ as a relation matrix. Express V as a direct sum of cyclic groups.

(12) 10. Let V be an abelian group generated by elements a, b, c. Assume that $2a = 6b, 2b = 6c, 2c = 6a$, and that these three relations generate all the relations on a, b, c.

(i) Write down a relation matrix for V.

(ii) Find generators x, y, z for V such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by x, y, z. Express your generators x, y, z in terms of a, b, c. What is the order of V?

(8) 11. List up to isomorphism all abelian groups of order 16.

(6) 12. Let F be a field.

(i) What is the dimension of the vector space of all 3-linear functions $D : F^{3\times 3} \to F$?

(ii) What is the dimension of the vector space of all 3-linear alternating functions $D : F^{3\times 3} \to F$?

(12) 13. Prove that a linear operator $T : \mathbb{R}^5 \to \mathbb{R}^5$ has a cyclic vector if and only if every linear operator $S : \mathbb{R}^5 \to \mathbb{R}^5$ that commutes with T is a polynomial in T.

(16) 14. Assume that V is a finite-dimensional vector space over an infinite field F and $T : V \to V$ is a linear operator. Give to V the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(i) Outline a proof that V is a direct sum of cyclic $F[x]$-modules.

(ii) In terms of the expression for V as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that V have only finitely many T-invariant F-subspaces? Explain.

(14) 15. Let M be a module over the integral domain D. Recall that a submodule N of M is said to be *pure* if the following holds: whenever $y \in N$ and $a \in D$ are such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(i) If N is a direct summand of M, prove that N is pure in M

(ii) For $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of x in the quotient module M/N. If N is a pure submodule of M and ann $\overline{x}$ $= \{a \in D \,|\, a\overline{x} = 0\}$ is a principal ideal (d) of D, prove that there exists $x' \in M$ such that $x + N = x' + N$ and ann $x' = \{a \in D \,|\, ax' = 0\}$ is the principal ideal (d).

(12) 16 Let M be a finitely generated module over the polynomial ring $F[x]$, where F is a field, and let N be a pure submodule of M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.