MATH 554
August 2008

Instructions: Give a complete solution to each question. For problems with multiple parts you may assume the result of the previous parts to solve the subsequent parts. Begin each problem on a new sheet of paper. Be sure your name is on every sheet of your solutions

Notation: The following are standard for this examination. If R is a ring, $M_n(R)$ is the collection of $n \times n$ matrices with R-entries, and $R[x]$ is the ring of plynomials with R–coefficients. The symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. The symbol I_n denotes the $n \times n$ identity matrix, and I_V is the identity transformation of a vector space V.

1. (10 points) Let R be a principal ideal domain. A finitely generated R-module M is said to be **indecomposable** if no submodule of M is a direct summand of M, i.e., it is impossible to find proper submodules M_1, M_2 of M so that $M = M_1 \oplus M_2$. Determine all indecomposable R–modules.

2. Let R be a commutative ring with identity 1_R.

(a) **(6 points)** Suppose $A \in M_n(R)$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$.
Show that, for each i, $b_i \det A = 0$.

(b) **(3 points)** Use (a) to show that if R is an integral domain and $A \in M_n(R)$ is singular (i.e., the kernel of A is a non-zero submodule of R^n) then $\det A = 0$.

3. (10 points) Suppose $A \in M_9(\mathbb{C})$, and $I = I_9$ satisfy the following conditions:

i) $\text{rank}(A + 2I) = 8$, and $\text{rank}(A + 2I)^k = 7$, for $k \geq 2$;
ii) $\text{rank}(A - (2i)I) = 7$, and $\text{rank}(A - (2i)I)^k = 6$, for $k \geq 2$;
iii) $\text{rank}(A-3I) = 8, \text{rank}(A-3I)^2 = 7, \text{rank}(A-3I)^3 = 6$, and $\text{rank}(A-3I)^k = 5$, for $k \geq 4$.

Find the Jordan Canonical form of A.

4. Let V be a real or complex inner product space, with given inner product $(\, , \,)$.

(a) **(4 points)**Prove that any collection of non-zero orthogonal vectors in V is linearly independent.

(b) **(5 points)** Let $\{v_1, v_2, \ldots, v_n\}$ be an orthogonal subset of V. Prove that, for any $w \in V$,

$$||w||^2 \geq \sum_{i=1}^{n} \frac{|(w, v_i)|^2}{||v_i||^2}.$$

5. (8 points) Let G be a group (not necessarily abelian). Suppose $\rho : G \to GL_n(\mathbb{C})$ is a homomorphism, i.e., $\rho(g) : \mathbb{C}^n \to \mathbb{C}^n$ is linear for each $g \in G$, and $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$, for all $g_1, g_2 \in G$. Finally, suppose the only G–invariant subspaces are $\{0\}$ and $\mathbb{C}^n$,, i.e., if W is a subspace of $\mathbb{C}^n$ and $\rho(g)W \subset W$ for all $g \in G$, then $W = \{0\}$ or $\mathbb{C}^n$. Show that if $A \in M_n(\mathbb{C})$ satisfies $A\rho(g) = \rho(g)A$ for all $g \in G$, then $A = cI_n$, for some $c \in \mathbb{C}$.
Hint: Find some G–invariant subspaces associated with A.

6. (12 points) Find the characteristic polynomial, minimal polynomial, and rational canonical form of the matrix

$$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix} \in M_4(\mathbb{Q}).$$

7. (3 points) Suppose T is a linear operator on $\mathbb{R}^n$, $f \in \mathbb{R}[x]$ and α is a (real) eigenvalue of $f(T)$. Is there a (real) eigenvalue β of T so that $f(\beta) = \alpha$? Give a proof or counterexample.

8. (5 points each) Let F be a field with p elements.

(a) Determine the order of the group $GL_3(\mathbb{F})$ of 3×3 invertible matrices with entries in F.
(b) Determine the order of the group $SL_3(F)$, the elements of $GL_3(F)$ of determinant 1.

9. (4 points each) Let V be a finite dimensional complex inner product space, and suppose T is a normal operator on V.

(a) Prove T is self adjoint if and only if all eigenvalues of T are real.
(b) Prove T is unitary if and only if all eigenvalues of T have norm 1.

10. (5 points each) Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Note that S^1 is an abelian group with the operation of complex multiplication.

(a) Is S^1 finitely generated? Why or why not?
(b) Let $\chi : S^1 \to GL_1(\mathbb{C}) \simeq \mathbb{C} \setminus \{0\}$. be a $\mathbb{Z}$–linear map (i.e., a group homomorphism). Suppose $z_1, z_2, \ldots, z_k$ are elements of S^1 of finite orders $m_1, m_2, \ldots, m_k$, and further suppose $\gcd(m_i, m_j) = 1$, for $i \neq j$. Show there is a positive integer n so that $\chi(z_i) = z_i^n$ for $i = 1, 2, \ldots, k$.

11. (5 points) Let F be a field and $t_0, t_1, \ldots, t_n$ be distinct elements of F. Given elements $a_0, a_1, \ldots, a_n \in F$, show there is a polynomial $f \in F[x]$, with $\deg f \leq n$, so that $f(t_i) = a_i$, for $i = 0, 1, \ldots, n$.

12. (6 points) Let F be a field, $A \in M_n(F)$, and let $T_A : M_n(F) \to M_n(F)$ be given by $T_A(B) = AB$. Show the minimal polynomial of T_A is the minimal polynomial of A. Are the characteristic polynomials of A and T_A equal as well?