Math 554 Qualifying Exam. August, 2009 (Jiu-Kang Yu)

1. Let J_e be the $e \times e$ *complex* matrix with $J_{j+1,j} = 1$ for $j = 1, \ldots, e-1$, $J_{i,j} = 0$ if $i \neq j+1$. It is the so-called $e \times e$ nilpotent Jordan block.

Let $e_1 \geq \cdots \geq e_r$ be a decreasing sequence of positive integers and let $A = J_{e_1,\ldots,e_r}$ be the direct sum of $J_{e_1}, \ldots, J_{e_r}$.

(a) (8 points) Compute $\dim \ker A^m$, for $m \geq 0$.

(b) (8 points) Show, without using the structure theorem, that if $J_{e_1,\ldots,e_r}$ is similar to $J_{f_1,\ldots,f_s}$ (where $f_1 \geq \cdots \geq f_s$ is another decreasing sequence of positive integers), then $r = s$ and $e_i = f_i$ for $i = 1, \ldots, r$.

(c) (8 points) What is the Jordan form of A^2? It is enough to describe the sizes (and eigenvalues) of its Jordan blocks.

(d) (8 points) What is the Jordan form of $A^2 + A$?

2. (10 points) Let $\mathbb{F}_2$ be the finite field $\mathbb{Z}/2\mathbb{Z}$. How many similarity classes of 3×3 invertible matrices over $\mathbb{F}_2$ are there? You may use the fact that there are 2,1,2 monic irreducible polynomial of degree $1, 2, 3$ respectively. It may help to consider the rational canonical forms.

3. Let $V = M_{2\times 2}(\mathbb{R})$ be the real vector space of 2×2 real matrices. Define two functions $q_1, q_2 : V \to \mathbb{R}$ by $q_1(A) = \mathrm{Tr}(A^2)$, $q_2(A) = \mathrm{Tr}(A.A^t)$, where $\mathrm{Tr}(B)$ is the trace of B and A^t is the transpose of A.

(a) (8 points) Show that q_1 and q_2 are quadratic forms.

(b) (8 points) What is the signature of q_1?

(c) (8 points) What is the signature of q_2?

Recall that the signature of q is (r, s) if r (resp. s) is the number of positive (resp. negative) entries in the diagonal when q is represented by a diagonal matrix.

4. Let $A = \begin{pmatrix} 2 & -4 & 6 \\ -2 & 10 & -12 \\ 4 & -14 & 18 \end{pmatrix}$.

(a) (8 points) Find the Smith normal form of A as a matrix over $\mathbb{Z}$. That is, find integers $d_1|d_2|d_3$ such that there exist $U, V \in \mathrm{GL}_3(\mathbb{Z})$ satisfying $UAV = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$.

(b) (8 points) Consider the homomorphism $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ defined by $f(x) = A.x$. Describe $\mathbb{Z}^3/f(\mathbb{Z}^3)$ as a direct sum of cyclic groups.

5. (8 points) Let $A.\vec{x} = \vec{b}$ be a system of n equations in m variables, where A is an $n \times m$ matrix with entries in $\mathbb{Q}$. Show that if the system has a solution in $\mathbb{C}^m$, then it has a solution in $\mathbb{Q}^m$.

6. (10 points) Let U be an $n \times n$ unitary matrix such that $I_n - U$ is invertible. Show that $A = (I + U)/(I - U)$ satisfies $A^* = -A$, where A^* is the conjugate transpose of A. Show that the eigenvalues of U are complex numbers λ satisfying $|\lambda| = 1$. What can you say about the eigenvalues of A?