Math 554 Qualifying Examination

January 2009 – J. Lipman

- *Each question 1–5 is worth 10 points (50 total).*
- *In answering any part of a question you may assume the preceding parts.*
- *Please begin your answer to each of the six questions on a new sheet of paper.*

1. (a) Let $0 \to U \to V \to W \to 0$ be an exact sequence of finite-dimensional vector spaces. Prove that $\dim U + \dim W = \dim V$.

(b) Prove: for subspaces W_1 and W_2 of a finite-dimensional vector space W,

$$\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2).$$

2. Let k be a field, and let M_n denote the k-vector space of $n \times n$ matrices with entries in k. For any $A \in M_n$, define $T_A\colon M_n \to M_n$ by $T_A(B) = AB \ \ (B \in M_n)$.

(a) Prove that if B is in the kernel of T_A then $\operatorname{rank} B \le n - \operatorname{rank} A$.

(b) Let E_A be the $k[X]$-module associated with the linear map $t_A\colon k^n \to k^n$ given by left multiplication by A (where the elements of k^n are viewed as $n \times 1$ column vectors), and let E_{T_A} be the $k[X]$-module associated with the linear map T_A. Prove that E_{T_A} is isomorphic to a direct sum of n copies of E_A.

(c) Prove: $AB = BA$ if and only if t_B is a $k[X]$-module map of E_A into itself.

3. (a) Find the Jordan normal form for the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

(b) Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the linear map associated to the matrix given in (a). How many T-invariant subspaces are there in $\mathbb{C}^3$?

(c) Find the invariant factors of the $\mathbb{C}[X]$-module associated to the linear map given by the matrix in (a).

4. (a) Describe carefully what is meant by the adjoint $\operatorname{Adj}(A)$ of an $n \times n$ matrix with entries in a commutative ring R.

This is not the same as the adjoint of an operator on an inner-product space!

(b) Express the determinant $\det(\operatorname{Adj}(A))$ as a function of $\det(A)$. Justify your answer, briefly.

5. Let V be a finite-dimensional $\mathbb{C}$-vector space, with a given positive definite hermitian form. Recall that an operator (= $\mathbb{C}$-linear map) $A\colon V \to V$ is normal if and only if it has a *spectral decomposition*: $A = \sum_{i=1}^{n} \alpha_i E_i$ where the $\alpha_i \in \mathbb{C}$ are distinct and the E_i are orthogonal projections (i.e., idempotent hermitian operators) such that $\sum_i E_i = \mathbf{1}$ (the identity map), and $E_i E_j = 0$ whenever $i \neq j$.

Let $A\colon V \to V$ and $B\colon V \to V$ be normal operators, with respective spectral decompositions

$$A = \sum_{i=1}^{n} \alpha_i E_i, \qquad B = \sum_{j=1}^{m} \beta_j F_j.$$

(a) Prove that

$$BA = AB \iff E_i F_j = F_j E_i \text{ for all } i, j.$$

(b) Suppose that $BA = AB$, let $h\colon \mathbb{C}^2 \to \mathbb{C}$ be any function such that the mn numbers $h(\alpha_i, \beta_j)$ are distinct, and define

$$(*) \qquad H = h(A, B) := \sum_{i,j} h(\alpha_i, \beta_j) E_i F_j.$$

Prove:

(i) $(*)$ is a spectral decomposition of H.

(ii) There exist polynomials $f(T), g(T) \in \mathbb{C}[T]$ such that $A = f(H)$, $B = g(H)$.