Math 55400 **Qualifying Exam** **August, 2010** **W. Heinzer**

PUID: ____

Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	20	
4	14	
5	16	
6	14	
7	16	
8	20	
9	14	
10	12	
11	14	
12	12	
13	16	
14	12	
Total	200	

1. (20 pts) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V and let $R = T(V)$ denote the range of T.

(a) Prove that R has a complementary T-invariant subspace if and only if R is independent of the null space N of T, i.e., $R \cap N = 0$.

(b) If R and N are independent, prove that N is the unique T-invariant subspace of V that is complementary to R.

2. (20 pts) Let V be a 5-dimensional vector space over a field F and let $T : V \to V$ be a linear operator.

(a) Prove that V is the direct sum of its two subspaces $\operatorname{Ker} T^5$ = the null space of T^5 and $\operatorname{Im} T^5 = T^5(V)$, the range of T^5.

(b) Give an example of a linear operator T such that V is not the direct sum of its subspaces $\operatorname{Ker} T$ and $\operatorname{Im} T$.

3. (14 pts) Let n be a positive integer, let V be an n-dimensional vector space over a field and let $T : V \to V$ be a linear operator. Prove or disprove that

$$\operatorname{rank} T \; + \; \operatorname{rank} T^3 \; \geq \; 2 \operatorname{rank} T^2.$$

4. (16 pts) Let F be a field of characteristic zero and let V be a finite-dimensional vector space over F. Recall that a linear operator $E : V \to V$ is a *projection operator* on V if $E^2 = E$. Assume that $E_1, \dots, E_k$ are projection operators on V and that $E_1 + \cdots + E_k = I$, the identity operator on V. Prove that $E_i E_j = 0$ for $i \neq j$.

5. (14 pts) Classify up to similarity all 3×3 complex matrices A such that $A^3 = I$, the identity matrix. How many equivalence classes are there?

6. (16 pts) Let V be a finite-dimensional complex inner product space and let $T : V \to V$ be a linear operator. Prove that T is self-adjoint if and only if $\langle T\alpha, \alpha \rangle$ is real for every $\alpha \in V$.

7. (20 pts) Let p be a prime integer and let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let V be a vector space over F and $T : V \to V$ a linear operator. Assume that T has characteristic polynomial x^3 and minimal polynomial x^2.

(a) Express V as a direct sum of cyclic $F[x]$-modules.

(b) How many 1-dimensional T-invariant subspaces does V have?

(c) How many of the 1-dimensional T-invariant subspaces of V are direct summands of V?

(d) How many 2-dimensional T-invariant subspaces does V have?

(e) How many of the 2-dimensional T-invariant subspaces of V are direct summands of V?

8. (14 pts) Let M be a module over the integral domain D. Recall that a submodule N of M is said to be *pure* if the following holds: whenever $y \in N$ and $a \in D$ are such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(a) If N is a direct summand of M, prove that N is pure in M.

(b) For $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of x in the quotient module M/N. If N is a pure submodule of M, and $\operatorname{ann} \overline{x} = \{a \in D \mid a\overline{x} = 0\}$ is the principal ideal (d) of D, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\operatorname{ann} x' = \{a \in D \mid ax' = 0\}$ is the principal ideal (d).

9. (12 pts) Let M be a finitely generated module over the polynomial ring $F[x]$, where F is a field, and let N be a pure submodule of M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.

10. (14 pts) Let D be a principal ideal domain, let n be a positive integer, and let $D^{(n)}$ denote a free D-module of rank n.

(a) If L is a submodule of $D^{(n)}$, prove that L is a free D-module of rank $m \leq n$.

(b) If L is a proper submodule of $D^{(n)}$, prove or disprove that the rank of L must be less than n.

11. (7 pts) Let V be a 5-dimensional vector space over the field F and let $T : V \to V$ be a linear operator such that $\operatorname{rank} T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of T. Explain.

12. (5 pts) List up to isomorphism all abelian groups of order 24.

13. (16 pts) Let V be an abelian group generated by elements a, b, c. Assume that $2a = 4b, 2b = 4c, 2c = 4a$, and that these three relations generate all the relations on a, b, c.

(a) Write down a relation matrix for V.

(b) Find generators x, y, z for V such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of the cyclic subgroups generated by x, y, z.

(c) Express your generators x, y, z in terms of a, b, c.

(d) What is the order of V?

14. (4 pts) State true or false and justify: If N_1 and N_2 are 4×4 nilpotent matrices over the field F and if N_1 and N_2 have the same minimal polynomial, then N_1 and N_2 are similar.

15. (4 pts) State true or false and justify: If A and B are $n \times n$ matrices over a field F, then AB and BA have the same minimal polynomial.

16. (4 pts) State true or false and justify: If V is a finite-dimensional vector space and W_1 and W_2 are subspaces of V such that $V = W_1 \oplus W_2$, then for any subspace W of V we have $W = (W \cap W_1) \oplus (W \cap W_2)$.