Math 55400 **Qualifying Exam** **August, 2011** **W. Heinzer**

PUID: ____

Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	18	
4	18	
5	16	
6	16	
7	17	
8	16	
9	20	
10	14	
11	16	
12	15	
13	14	
Total	200	

Notation: Let F be a field, let n be a positive integer, and let V be an n-dimensional vector space over F. Let S and T be linear operators on V.

1. (13 pts) If T has n distinct characteristic values and S commutes with T, prove that there exists a polynomial $f(t) \in F[t]$ such that $S = f(T)$.

2. (7 pts) Prove or disprove: If S commutes with T and $a \in F$, then the null space of $T - aI$ is invariant for S.

Notation: If K is a commutative ring and m and n are positive integers, then $K^{m\times n}$ denotes the K-module of $m \times n$ matrices with entries in K.

3. (6 pts) State true or false and justify: If $\mathcal{F} \subset \mathbb{C}^{4\times 4}$ is a subspace of commuting matrices, then $\dim \mathcal{F} \leq 4$.

4. (12 pts) Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(a) Write down a relation matrix for V as a $\mathbb{Z}$-module.

(b) Let W be the cyclic subgroup of V generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$. Write down a relation matrix for W.

(c) Write down a relation matrix for the quotient module V/W.

5. Let K be a commutative ring with identity, n a positive integer, and let $D : K^{n\times n} \to K$ be a function.

(a) (3 pts) Define "D is n-linear".

(b) (3 pts) If D is n-linear, define "D is alternating".

(c) (3 pts) Define "D is a determinant function."

(d) (4 pts) If $n = 3$ and K is a field, what is the dimension of the K-vector space of all 3-linear functions on $K^{3\times 3}$?

(e) (5 pts) If K is the polynomial ring $\mathbb{Q}[\{x_{ij}\}]$, where $1 \le i \le 5$, $1 \le j \le 5$, and $A = (x_{ij}) \in K^{5\times 5}$, then $\det A$ is a sum of monomials in the x_{ij}. How many terms are in this sum? Explain.

6. (8 pts) Let V be an n-dimensional vector space over the field F and let $T : V \to V$ be a linear operator. Assume that $c \in F$ is such that there exists a nonzero vector α with $T\alpha = c\alpha$. Prove that there exists a nonzero linear functional f on V such that $T^t f = cf$, where T^t is the transpose of T.

7. (8 pts) Let F be a field and let L be a linear functional on the polynomial ring $F[x]$ having the property that $L(fg) = L(f)L(g)$ for all polynomials $f, g \in F[x]$. Prove that either $L = 0$ or there exists $c \in F$ such that $L(f) = f(c)$ for all $f \in F[x]$.

8. Let V be a finite-dimensional vector space over a field F, let $T : V \to V$ be a linear operator, and let $p(x) \in F[x]$ be the minimal polynomial of T. Assume that $p(x) = p_1^{r_1} \cdots p_k^{r_k}$, where the $p_i \in F[x]$ are distinct monic irreducible polynomials, $i = 1, \cdots, k$, and the r_i are positive integers. Let $W_i = \{v \in V \mid p_i(T)^{r_i}(v) = 0\}$.

(a) (8 pts) Describe how to obtain linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each i, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on V.

(b) (8 pts) If $p(x)$ is a product of linear polynomials, describe how to obtain a diagonalizable operator D and a nilpotent operator N such that $T = D + N$, where D and N are both polynomials in T.

9. (8 pts) Prove or disprove: if V is a vector space over a field F and $T : V \to V$ is a linear operator such that every subspace of V is invariant under T, then T is a scalar multiple of the identity operator.

10. Let F be a field and let $g(x) \in F[x]$ be a monic polynomial.

(a) (4 pts) Describe the $F[x]$-submodules of $V = F[x]/(g(x))$.

(b) (5 pts) If $g(x) = x^3(x-1)$, diagram the lattice of $F[x]$-submodules of $V = F[x]/(g(x))$.

11. (16 pts) Let D be a principal ideal domain and let V and W denote free D-modules of rank 3 and 4, respectively. Assume that $\phi : V \to W$ is a D-module homomorphism, and that $\mathbf{B} = \{v_1, v_2, v_3\}$ is an ordered basis of V and $\mathbf{B}' = \{w_1, w_2, w_3, w_4\}$ is an ordered basis of W.

(a) Define what is meant by the coordinate vector of $v \in V$ with respect to the basis $\mathbf{B}$.

(b) Describe how to obtain a matrix $A \in D^{4\times 3}$ so that left multiplication by A on D^3 represents $\phi : V \to W$ with respect to $\mathbf{B}$ and $\mathbf{B}'$.

(c) How does the matrix A change if we change the basis $\mathbf{B}'$ by replacing w_2 by $w_2 + aw_1$ for some $a \in D$?

(d) How does the matrix A change if we change the basis $\mathbf{B}$ by replacing v_2 by $v_2 + av_1$ for some $a \in D$?

12. (20 pts) Let p be a prime integer and let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let V be a vector space over F and $T : V \to V$ a linear operator. Assume that T has characteristic polynomial x^4 and minimal polynomial x^3.

(a) Express V as a direct sum of cyclic $F[x]$-modules.

(b) How many 3-dimensional cyclic T-invariant subspaces does V have?

(c) How many of the 3-dimensional cyclic T-invariant subspaces of V are direct summands of V?

(d) How many noncyclic 3-dimensional T-invariant subspaces does V have?

(e) How many of the noncyclic 3-dimensional T-invariant subspaces of V are direct summands of V?

13. (14 pts) Let V be an abelian group generated by elements a, b, c. Assume that $3a = 6b$, $3b = 6c$, $3c = 6a$, and that these three relations generate all the relations on a, b, c.

(a) What is the order of V? Justify your answer.

(b) What is the order of the element a? Justify your answer

14. (10 pts) Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many of its proper subspaces.

15. (6 pts) Let F be a finite field with $|F| = q$, and let $G = \{A \in F^{3\times 3} \mid \det A \neq 0 \}$.

(a) What is $|G|$?

(b) Let $H = \{A \in G \mid \det A = 1\}$. What is $|H|$?

16. (6 pts) Let $A \in \mathbb{R}^{n \times n}$ and let $f_1, \ldots, f_n$ be the diagonal entries in the normal form of $xI - A$.

(i) For which matrices A is $f_1 \neq 1$?

(ii) For which matrices A is $f_{n-1} = 1$?

17. (9 pts) Let $A \in \mathbb{R}^{3 \times 3}$ be such that $\det A = 3$ and let $\operatorname{adj}(A) \in \mathbb{R}^{3 \times 3}$ denote the classical adjoint of A.

(a) What is the product $\operatorname{adj}(A)A$?

(b) What is $\det(\operatorname{adj} A)$?

(c) What is $\operatorname{adj}(\operatorname{adj} A)$?

18. (6 pts) Let V be a 4-dimensional vector space over the field F and let $T : V \to V$ be a linear operator such that $\operatorname{rank} T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of T. Explain.

19. (8 pts) Let F be a field and let $V = F^{4\times4}$. Let W be the subspace of V spanned by all matrices of the form $C = AB - BA$, where $A, B \in V$. Prove that W is the subspace of V of matrices having trace zero.