LINEAR ALGEBRA COMPREHENSIVE EXAM – JAN, 2011

Attempt all questions. Time 2 hrs

(1) Let C be a commutative ring with identity, E be a finitely generated projective C-module, and $u \in \mathrm{End}_C(E)$.
   (a) (2 pts) Define $\mathrm{Tr}(u)$ (trace of u).
   (b) (8 pts) Let F be another finitely generated projective module and $v : E \to F$, and $w : F \to E$ be two linear maps. Prove that
$$\mathrm{Tr}(v \circ w) = \mathrm{Tr}(w \circ v).$$

(2) Let L be a free module over a principal ideal domain A, and let M be a submodule of finite rank n.
   (a) (2 pts) Given $x \in L$ define the content, $\mathfrak{c}_L(x)$, of x.
   (b) (10 pts) Prove that there exists a basis B of L, and n elements $e_1, \ldots, e_n$ of B, and corresponding elements $\alpha_1, \ldots, \alpha_n$ of A such that:
      (i) $\alpha_1 e_1, \ldots, \alpha_n e_n$ form a basis of M;
      (ii) α_i divides α_{i+1} for $1 \leq i \leq n-1$.
   (c) (8 pts) Prove that every finitely generated module E over a principal ideal domain A is a direct sum of a finite number of cyclic modules.

(3) Let k be a field and V a finite dimensional k-vector space, and $u \in \mathrm{End}_k(V)$. Let $V[X] = k[X] \otimes_k V$.
   (a) (2 pts) Define the $k[X]$-module V_u. Which subspaces of V are sub-modules of V_u ?
   (b) (2 pts) Prove that there exists a linear map $\phi : V[X] \to V_u$, such that for every $p \in k[X]$ and $v \in V$, $\phi(p \otimes v) = p(u) \cdot v$.
   (c) (6 pts) Prove that the following sequence
$$V[X] \xrightarrow{X-\bar{u}} V[X] \xrightarrow{\phi} V_u \to 0$$
   is exact.
   (d) (10 pts) Recall that the characteristic polynomial $\chi_u \in k[X]$ is defined to be the determinant of the endomorphism $X - \bar{u}$ of the free $k[X]$-module $V[X]$. Prove Cayley-Hamilton's theorem, namely that $\chi_u(u) = 0$ in $\mathrm{End}_k(V)$.

(4) Let k be a field and V be a finite dimensional k vector space, and A, B two endomorphisms of V.

(a) (8 pts) Prove or disprove that A is diagonalizable if and only if the minimal polynomial of A equals its characteristic polynomial.
(b) (6 pts) Suppose that A, B commute. Prove that each eigenspace of A is closed under B.
(c) (2 pts) What does it mean to say that A and B are simultaneously diagonalizable ?
(d) (10 pts) Prove that A, B are simultaneously diagonalizable if and only if both A and B are diagonalizable and $AB = BA$.
(e) (4 pts) Let $k = \mathbb{C}$ and $V = \mathbb{C}^2$. Give an example of an endomorphism of V that is not diagonalizable.

(5) Let V be a complex inner product space and T an endomorphism of V.
(a) (2 pts) Define (when it exists) the adjoint, T^* of T.
(b) (4 pts) Prove that if T is self-adjoint and α is an eigenvalue of T, then $\alpha \in \mathbb{R}$.
(c) (4 pts) Prove that if W is a subspace of V closed under T, then $W^\perp$ is closed under T^*.
(d) (10 pts) Prove that a finite dimensional complex inner product space has an orthogonal basis consisting of eigenvectors of T, if T is a self-adjoint endomorphism of V.