QUALIFYING EXAMINATION

January 2014

MA 554

1. (11 points) Let R be an integral domain. Recall that an R-module is called *torsionfree* if for every element $a \neq 0$ in R and $m \neq 0$ in the module, $am \neq 0$. Let N be an R-submodule of an R-module M.

(a) Show that if N and M/N are torsionfree, then so is M.

(b) Show that the converse holds in (a) for every N, M if and only if R is a field.

2. (11 points) Let K be a field and V a vector space over K. Let φ and ψ be K-endomorphisms of V so that $\varphi\psi = 0$ and $\mathrm{id}_V = \varphi + \psi$. Show that $V = \mathrm{im}(\varphi) \oplus \mathrm{im}(\psi)$.

3. (14 points) Let R be an integral domain and let A, B be n by n matrices with entries in R. Assume that $AB = aI_n$, where $a \neq 0$ is in R and I_n denotes the n by n identity matrix. Show that $AB = BA$.

4. (17 points) Up to $\mathbb{Q}[x]$-isomorphisms, determine all $\mathbb{Q}[x]$-modules M that are annihilated by the polynomial $x(x^3-2)^3$ and satisfy $\dim_{\mathbb{Q}} M = 8$. How many non-isomorphic $\mathbb{Q}[x]$-modules of this type are there?

5. (15 points) Over an arbitrary field, consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & -1 \\ -1 & 0 & 0 & -1 \\ -2 & 1 & 0 & -2 \end{pmatrix}.$$

Find the Jordan canonical form of A.

6. (15 points) Let A be a $2n$ by $2n$ matrix with entries in $\mathbb{R}$ satisfying $A^2 = -I_{2n}$. Show that A is similar to the matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

7. (17 points) Let V be a finite-dimensional inner product space over K, where $K = \mathbb{R}$ or $K = \mathbb{C}$. Let φ be a K-endomorphism of V and write φ^T for its adjoint.

(a) Show that $\ker(\varphi^T) = (\mathrm{im}(\varphi))^{\perp}$.

(a) Show that $\ker(\varphi) = (\mathrm{im}(\varphi))^{\perp}$ if φ is normal.

(b) Show that $\ker(\varphi^n) = \ker(\varphi)$ for every $n \geq 1$ if φ is normal.