Math 55400 **Qualifying Exam** **Jan. 8, 2016** **W. Heinzer**

Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	20	
3	20	
4	20	
5	20	
6	18	
7	20	
8	20	
9	16	
10	16	
11	15	
12	15	
Total	200	

1. (20 pts) Classify up to similarity all matrices $A \in \mathbb{Q}^{3\times 3}$ such that $A^3 = I$.

2. (20 pts) Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $T : V \to V$ be a linear operator.

(a) (2 pts) Define the adjoint T^* of T.

(b) (6 pts) If W is a T-invariant subspace of V, prove or disprove that the orthogonal complement $W^\perp$ is T^*-invariant.

(c) (6 pts) If $T = T^*$, prove that every characteristic value of T is a real number.

(d) (6 pts) Assume that $T = T^*$ and that c and d are distinct characteristic values of T. If α and β in V are such that $T\alpha = c\alpha$ and $T\beta = d\beta$, prove that α and β are orthogonal.

3. (12 pts) Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be a linear operator and let $g(x)$ be a polynomial in $\mathbb{C}[x]$. If c is a characteristic value for $g(T)$, must there exist a characteristic value a for T such that $g(a) = c$? Explain.

4. (8 pts) State true or false and justify your answer: If V is a finite-dimensional vector space and W_1 and W_2 are subspaces of V such that $V = W_1 \oplus W_2$, then for any subspace W of V we have $W = (W \cap W_1) \oplus (W \cap W_2)$.

5. Let $A \in \mathbb{C}^{3\times 3}$ be a diagonal matrix with main diagonal entries $1, 2, 3$. Define $T_A : \mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 3}$ by $T_A(B) = AB - BA$.

(a) (4 pts) What is the dimension of the null space of T_A?

(b) (4 pts) What is the dimension of the range of T_A?

(c) (4 pts) What are the characteristic values of T_A?

(d) (4 pts) What is the minimal polynomial of T_A?

(e) (4 pts) Is T_A diagonalizable? Explain.

6. (18 pts) Let D be a principal ideal domain and let V and W denote free D-modules of rank 3 and 2, respectively. Assume that $\varphi : V \to W$ is a D-module homomorphism, and that $\mathbf{B} = \{v_1, v_2, v_3\}$ is an ordered basis of V and $\mathbf{B}' = \{w_1, w_2\}$ is an ordered basis of W.

(a) (4 pts) Define the coordinate vector of $v \in V$ with respect to the basis $\mathbf{B}$.

(b) (4 pts) Describe how to obtain a matrix $A \in D^{2\times 3}$ so that left multiplication by A on D^3 represents $\varphi : V \to W$ with respect to $\mathbf{B}$ and $\mathbf{B}'$.

(c) (5 pts) How does the matrix A change if we change the basis $\mathbf{B}$ by replacing v_1 by $v_1 + av_2$ for some $a \in D$?

(d) (5 pts) How does the matrix A change if we change the basis $\mathbf{B}'$ by replacing w_1 by $w_1 + aw_2$ for some $a \in D$?

7. (20 pts) Let V be a 4-dimensional vector space over $\mathbb{C}$, and let $L(V,V)$ be the vector space of linear operators on V. Let $\mathcal{F}$ be a subspace of $L(V,V)$ such that for every $T, U \in \mathcal{F}$, we have $TU = UT$.

(a) (8 pts) Demonstrate with an example that it is possible for there to exist in $\mathcal{F}$ five elements that are linearly independent over $\mathbb{C}$.

(b) (12 pts) If there exists $T \in \mathcal{F}$ having at least two distinct characteristic values, prove or disprove that $\dim \mathcal{F} \leq 4$.

8. (20 pts) Let V be a finite-dimensional vector space over a field F and let $T : V \to V$ be a linear operator. Give to V the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(a) If $\{v_1, \cdots, v_n\}$ are generators for V as an $F[x]$-module, what does it mean for a matrix $A \in F[x]^{m\times n}$ to be a relation matrix for V with respect to $\{v_1, \ldots, v_n\}$?

(b) If $F = \mathbb{C}$ and $A = \begin{bmatrix} x^2(x-1)^2 & 0 & 0 \\ 0 & x(x-1)(x-2) & 0 \\ 0 & 0 & x(x-2)^2 \end{bmatrix}$ is a relation matrix for V with respect to $\{v_1, v_2, v_3\}$, list the invariant factors of V.

(c) With assumptions as in part (b), list the elementary divisors of V and describe the direct sum decomposition of V given by the primary decomposition theorem.

(d) With assumptions as in part (b), write the Jordan form of the operator T.

9. (16 pts) Let V be a finite-dimensional vector space over a field F and let $T : V \to V$ be a linear operator. Give to V the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(a) Outline a proof that $V = \frac{F[x]}{(d_1)} \oplus \cdots \oplus \frac{F[x]}{(d_r)}$, where $d_1, \ldots, d_r$ are monic polynomials such that d_k divides d_{k-1} for $2 \leq k \leq r$.

(b) Assume the field F is infinite. In terms of the expression for V as a direct sum of cyclic $F[x]$-modules as in part (a), what are necessary and sufficient conditions in order that V have only finitely many T-invariant subspaces? Explain.

10. (16 pts) Let M be a module over the integral domain D. A submodule N of M is *pure* in M if the following holds: whenever $y \in N$ and $a \in D$ are such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(a) (8 pts) For N a submodule and $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of x in the quotient module M/N. If N is pure in M, and $\operatorname{ann} \overline{x} = \{a \in D \mid a\overline{x} = 0\}$ is the principal ideal (d) of D, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\operatorname{ann} x' = \{a \in D \mid ax' = 0\}$ is the principal ideal (d).

(b) (8 pts) Let $M = \langle \alpha \rangle$ be a cyclic $\mathbb{Z}$-module of order 12. List the submodules of M and indicate which of these submodules are pure in M.

11. (15 pts) Let F be a field and let M be a finitely generated module over the polynomial ring $F[x]$. Let N be a submodule of M. If N is pure in M, prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.

12. (15 pts) Let $A \in \mathbb{C}^{4\times 4}$ be a diagonal matrix with exactly three distinct entries on its main diagonal.

(a) (5 pts) What is the dimension of the vector space over $\mathbb{C}$ of matrices that are polynomials in A?

(b) (5 pts) What is the dimension of the vector space over $\mathbb{C}$ of matrices $B \in \mathbb{C}^{4\times 4}$ such that $AB = BA$?

(c) (5 pts) If $B \in \mathbb{C}^{4\times 4}$ is a diagonal matrix with exactly three distinct entries on its main diagonal, is B similar to a polynomial in A? Justify your answer.